A RAPID-PRESSURE CORRELATION REPRESENTATION CONSISTENT WITH THE TAYLOR-PROUDMAN THEOREM MATERIALLY-FRAME-INDIFFERENT IN THE 2D LIMIT

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ABSTRACT

A nonlinear representation for the rapid-pressure correlation appearing in the Reynolds stress equations, consistent with the Taylor-Proudman theorem, is presented. The representation insures that the modeled second-order equations are frame-invariant with respect to rotation when the flow is two-dimensional in planes perpendicular to the axis of rotation. The representation satisfies realizability in a new way: a special *ansatz* is used to obtain, analytically, the values of coefficients valid away from the realizability limit: the model coefficients are functions of the state of the turbulence that are valid for all states of the mechanical turbulence attaining their constant limiting values only when the limit state is achieved. Utilization of all the mathematical constraints are not enough to specify all the coefficients in the model. The unspecified coefficients appear as free parameters which are used to insure that the representation is asymptotically consistent with the known equilibrium states of a homogeneous sheared turbulence. This is done by insuring that the modeled evolution equations have the same fixed points as those obtained from computer and laboratory experiments for the homogeneous shear. Results of computations of the homogeneous shear, with and without rotation, and with stabilizing and destabilizing curvature, are shown. Results are consistently better, in a wide class of flows which the model not been calibrated, than those obtained with other nonlinear models.

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1. Introduction

Most turbulence models are devised for use in inertial coordinate systems. *Ad hoc* changes are then made to reflect the unusual effects on turbulence of swirl, curved streamlines or rotation of the coordinate system: one makes the length scale depend on Richardson number or adds terms to the dissipation equation calibrated to observed behavior. This approach does not make use of the mathematical requirements that the dependent variables and their evolution equations must satisfy and leads to models that perform poorly when used in situations substantially different from the benchmark flows for which they have been calibrated. There is no reason a second-order modeling method cannot be successfully applied to a high Reynolds number rotating turbulence: no new unknown terms appear in the equations and, for moderate Rossby number, very little of the phenomenology on which the technique is based changes.

The effect of rotation on the second-order moments is felt through the rapid-pressure-velocity correlation and the Coriolis terms. The difficulty with the equations, as modeled presently, can be seen when the equations are transformed to a rotating coordinate system: they are not materially-frame-indifferent in the two-dimensional limit, Speziale (1981), Hide (1977). This phenomenon, predicted by the Taylor-Proudman theorem requires that when the velocity field is two-dimensionalized by rapid rotation, without components along the axis of rotation, the equations must be independent of rotation. The problem with the current modeled second-order equations results from the inability of the rapid-pressure correlation model to reflect the physics embodied in the Taylor-Proudman theorem. This is another form of realizability: one must obtain, for a bounded flow, in the limit of rapid rotation, a frame-indifferent turbulence. In such a turbulence the rapid-pressure correlation appearing on the right hand side of the second order moment evolution equations equals the Coriolis terms appearing on the left hand side leaving the equations frame-indifferent. A rapid-pressure model consistent with these facts, reflecting more of the information contained in the Navier-Stokes equations, is required.

As a general tool is being developed to compute a wide class of flows for which there may not be any well documented benchmark flows with which to calibrate coefficients, it is necessary to construct a model from first principles, incorporating more of the physics. In the present new representation for the rapid-pressure strain correlation the use of all constants that do not come from first principles have been minimized: This is done this by requiring the representation to have the proper behavior in five different limits: 1) frame invariant with respect to rotation when the eigenvalue of the Reynolds stress tensor, $< u_i u_j >$, along the axis of rotation vanishes, 2) the realizable limit in which an arbitrary eigenvalue of the Reynolds stress vanishes, 3) the joint-realizable limit in which an eigenvalue of the tensor $< \theta\theta >< u_i u_j > - < \theta u_i >< \theta u_j >$ vanishes, 4) the isotropic limit in which the anisotropy tensor, $b_{ij} = 0$, and 5) asymptotic consistency with a stationary state of the turbulence in which $D/Dt\ b_{ij} = 0$. Of these five principles the last one is a statement based on experimental observation rather than mathematical fact. It is tacitly asusmed that for a specific class of flows, to a reasonably suitable approximation, there exists an equilibrium state to which the flow relaxes upon the removal of any disturbing forces.

Assuring the proper frame-invariance of the modeled second moment equations is done by requiring that in the limit of a turbulence two-dimensional in planes perpendicular to the axes of rotation the rapid-pressure velocity correlations in the heat-flux and Reynolds stress equations satisfy the "geostrophic" constraints, Ristorcelli (1987) and Ristorcelli and Lumley (1991b):

$$\begin{aligned} \epsilon_{pqn}\Omega_n X_{ipqj} &= \epsilon_{qjn}\Omega_n < u_i u_q > \\ \epsilon_{pqn}\Omega_n X_{pqi} &= \epsilon_{qin}\Omega_n < u_q\theta > . \end{aligned}$$

The term geostrophic is borrowed from the meteorological literature where it is used to describe the low Rossby number balance between Coriolis and pressure forces in the evolution equations for large scale weather. The constraint, however, is independent of how the flow is two-dimensionalized and is, therefore, independent of Rossby number. The requirements of the Taylor-Proudman theorem are subsumed by the requirements of the principle of two-dimensional frame invariance (2DMFI) first put forward, in the context of turbulence modeling, by Speziale (1981). A flow that is two-dimensionalized due to stable stratification or magnetic forces is also frame-indifferent in the two-dimensional two-component limit and the rapid-pressure correlation must still satisfy the geostrophic constraint for arbitrary rotation.

The requirement of two-dimensional material-frame-invariance comes from first principles: it is a rigorous limit of the Navier-Stokes equations for the fluctuating velocity, Speziale (1990). The principle of 2DMFI seems to have been given short shrift. Reynolds (1989), in a rapid distortion theory analysis of a decaying turbulence, has shown that rotation has a distinct effect on the second invariant, II, of the anisotropy tensor, b_{ij}, and goes on further to show that the equations for the invariants using the current rapid-pressure models are independent of rotation. He concludes that such models, because they don't show an explicit dependence on the rotation in the evolution equations for the invariants, are inadequate for rotating flows. A reference to the evolution equations for the invariants, for example Speziale *et al.* (1991), shows that both the production and dissipation, which are functions of the rotation through their dependence on the Reynolds stresses, appear in the evolution equations for the invariants. The equations, except for the case of a decaying turbulence in which there are no production terms, are not independent of rotation.

Shih and Shabbir (1990), following Reynolds argument decide that since the rotation terms do not explicitly appear in the evolution equation for the invariants (which is true for any rapid-pressure model that is a function of symmetric tensors) that the use of a properly frame-indifferent model will not produce the proper turbulence behavior. This argument is specious: the same argument applied to the trace of the Reynolds stress, the kinetic energy of the turbulence, $q^2 =< u_p u_p >$, one would abandon refinement of rapid-pressure models because they do not affect the energy of the turbulence

- the evolution equation for q^2 being independent of the rapid-pressure modeling. However, different rapid-pressure models produce different Reynolds stresses and different turbulence energies in the same flows. Even if the models fails to account for some predictions of RDT for a restricted class of flows neglecting results that are rigorous consequences of the Navier-Stokes equations, Speziale (1990), are not justified. (It should be kept in mind that the RDT limit violates various equilibrium assumptions made in the mathematical development of the second-order modeling method). The 2DMFI constraint is relevant to flows other than rotating flows: any flows in which the largest scales of the motion are two-dimensionalized by strong body forces will come arbitrarily close to being rotationally frame invariant. A dramatic example of this might be the collapse of the wake in a stably stratified environment. Shih and Shabbir (1990) also cite Reynolds (1987) as having shown that the 2DMFI constraint does not provide any new information. Here it is found that the geostrophic constraint produces six additional linearly independent constraint equations for the rapid-pressure model.

The present article is a description of the derivation and validation of a rapid-pressure model that satisfies the principle of material-frame-indifference in the two-dimensional limit. The present derivation of the 2DMFI model has been given earlier, in departmental reports in Ristorcelli (1987, 1991) or Ristorcelli and Lumley (1991b). In those previous developments of the 2DMFI model several free parameters were, for simplicity, set to zero. In this paper the stationary points of the homogeneous shear, as reflected in the structural equilibrium assumption, $D/Dt\ b_{ij} = 0$ where $b_{ij} = < u_i u_j > / < u_p u_p > - 1/3\delta_{ij}$, are used to set the unknown free parameters occurring in the model. The algebraic equations describing the stationary states of the homogeneous shear along with the experimentally determined values of the stationary state are used as constraints to set the free parameters in the model. This insures that the fixed points of the modeled differential equations are the same as the fixed points of a particular physical flow. This calibration is done *without* sacrificing any of the mathematical principles built into the model.

The next section of this article defines the problem. A third section presents a derivation of the tensor polynomial model for the rapid-pressure correlation. A discussion of the constraints as well as a derivation of the new geostrophic constraint is given. A new *ansatz* for the rapid-pressure correlation produces a model valid away from both the geostrophic (two-dimensional in planes perpendicular to the axis of rotation) and realizability limits (two-dimensional in that one of the eigenvalues of the Reynolds stress tensor dissappears). The issues of realizability in the light of some recent work, Speziale (1993), are discussed. A subsequent section uses results from matrix algebra to collapse the model to a structure similar to other rapid-pressure models with some interesting differences. The model is seen to have the same tensor bases as the FLT model and in the limit of a planar mean flow it has the same tensor bases as the quasi-linear SSG model. The two subsequent sections show computations done with the model and compare the results to other models. Following this a general discussion of modeling issues, in the context of realizability and the use of stationary states as the only self-consistent asymptotic limit for calibrating turbulence models is given.

2. The rapid-pressure correlation

In an incompressible turbulence in a rotating coordinate system, with buoyancy effects in the Boussinesq approximation, the Reynolds stress equations have the following form:

$$\begin{aligned} D/Dt < u_i u_j > +2(\epsilon_{ikp} < u_p u_j > +\epsilon_{jkp} < u_p u_i >)\Omega_k Ro^{-1} = &\ + < \theta u_i > \beta_j + < \theta u_j > \beta_i \\ -[< u_j u_p > U_{i,p} + < u_i u_p > U_{j,p}] &\ - < u_i u_j u_p >_{,p} \\ -[< p_{,j}\, u_i > + < p_{,i}\, u_j >] + Re^{-1} < u_i u_j >_{,pp} - &\ 2Re^{-1} < u_{i,p}\, u_{j,p} > \end{aligned}$$

$$\begin{aligned} D/Dt < \theta u_i > +2\epsilon_{pik}\Omega_k < \theta u_p > Ro^{-1} = &\ -[< \theta u_j > U_{i,j} + < u_i u_j > T_{,j}] + < \theta\theta > \beta_i \\ &\ - < \theta u_i u_j >_{,j} - < p_{,i}\theta > \\ &\ + Re^{-1}(1 + Pr^{-1})(< \theta u_i >_{,jj} -2 < \theta_{,j} u_{i,j} > . \end{aligned}$$

The velocity has been normalized by a characteristic velocity u_c and the Rossby number is $Ro = u_c/\Omega R_c$ where R_c is a length scale and Ω the rotation rate of the frame of reference. The gravity and rotation vectors are aligned with the 3 axis. Our concern is with the pressure-velocity and -temperature correlations, $< p_{,j}\, u_i >$ and $< p_{,i}\, \theta >$. An equation for the pressure fluctuations comes from the divergence of the Navier-Stokes equations for the fluctuating velocity

$$u_{i,t} + u_j U_{i,j} + U_j u_{i,j} + u_j u_{i,j} - < u_i u_j >_{,j} + 2\epsilon_{ikp}\Omega_k u_p Ro^{-1} = -p_{,i} + \theta\beta_i + Re^{-1} u_{i,jj}$$

which produces a Poisson equation for fluctuating pressure. The standard linear decomposition recognizes three terms

$$\begin{aligned} -p^r_{,ii} = &\ 2[U_{i,p} + \epsilon_{pik}\Omega_k Ro^{-1}] u_{p,i} \\ -p^s_{,ii} = &\ u_{i,j}\, u_{j,i} - < u_{i,j}\, u_{j,i} > \\ p^b_{,ii} = &\ \beta_i \theta_{,i} \end{aligned}$$

where p^r, p^s, p^b are respectively the rapid-pressure, the slow or return to isotropy pressure, and the buoyancy-pressure. The effects of rotation are felt through the rapid-pressure, p^r. Solution of the Poisson equation for the rapid-pressure is by application of Green's theorem

$$\phi(\mathbf{x}) = -\, (4\pi)^{-1} \int \phi(\mathbf{x}')_{,jj} d\mathbf{x}'/(\mathbf{x} - \mathbf{x}').$$

It is the moments of the solution that are required to close the second-order equations. For a homogeneous mean field, more than an integral scale away from any surfaces, a staightforward interchange of the order integration and averaging produces:

$$\begin{aligned} < p^r{}_{,j}\, u_i > + < p^r{}_{,i}\, u_j > = &\ -2[U_{q,p} + \epsilon_{pqk}\Omega_k Ro^{-1}][X_{ipqj} + X_{jpqi}] \\ < p^r{}_{,i}\, \theta > = &\ -2[U_{q,p} + \epsilon_{pqk}\Omega_k Ro^{-1}] X_{piq} \end{aligned}$$

where

$$\begin{aligned} X_{piq} = &\ (4\pi)^{-1} \int < \theta(\mathbf{x}) u_p(\mathbf{x}') >_{,i'q'} d\mathbf{x}'/(\mathbf{x} - \mathbf{x}') \\ X_{ipqj} = &\ (4\pi)^{-1} \int < u_i(\mathbf{x}) u_p(\mathbf{x}') >_{,j'q'} d\mathbf{x}'/(\mathbf{x} - \mathbf{x}'). \end{aligned}$$

The construction of a tensor polynomial model for the volume integrals of the two-point correlations, X_{ijkl} and X_{ijk}, is the subject of this article. The rapid-pressure covariance integral appearing in the heat flux equations, X_{ijk}, is treated as the Reynolds stresses and the heat fluxes are linked through Cauchy-Schwarz inequalities, $<\theta\theta><u_i u_j> - <\theta u_i><\theta u_j> \geq 0$ and their modeling cannot be done independently. The rapid-pressure term appearing in the Reynolds stress equations necessary to treat the mechanical turbulence problem is the focus of this article.

3. Obtaining a representation for the rapid-pressure correlation

A constitutive relation in which the integral of the two-point covariance is parameterized by a *local* function of the anisotropy tensor and heat-flux vector is proposed. The most general forms of such relationships are the following tensor polynomials

$$
\begin{aligned}
X_{ijkl}/<u_p u_p> = \;& A_1\delta_{ij}\delta_{kl} + A_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\
+\;& A_3\delta_{ij}b_{kl} + A_4 b_{ij}\delta_{kl} + A_5(b_{ik}\delta_{jl} + b_{il}\delta_{jk} + \delta_{ik}b_{jl} + \delta_{il}b_{jk}) \\
+\;& A_6\delta_{ij}b^2_{kl} + A_7 b^2_{ij}\delta_{kl} + A_8(b^2_{ik}\delta_{jl} + b^2_{il}\delta_{jk} + \delta_{ik}b^2_{jl} + \delta_{il}b^2_{jk}) \\
+\;& A_9 b_{ij}b_{kl} + A_{10}(b_{ik}b_{jl} + b_{il}b_{jk}) \\
+\;& A_{11}b_{ij}b^2_{kl} + A_{12}b^2_{ij}b_{kl} + A_{13}(b^2_{ik}b_{jl} + b^2_{il}b_{jk} + b_{ik}b^2_{jl} + b_{il}b^2_{jk}) \\
+\;& A_{14}b^2_{ij}b^2_{kl} + A_{15}(b^2_{ik}b^2_{jl} + b^2_{il}b^2_{jk})
\end{aligned}
$$

$$
\begin{aligned}
X_{pkj} = \;& D_1 <\theta u_p> \delta_{kj} + D_2(<\theta u_k>\delta_{pj} + <\theta u_j>\delta_{pk}) \\
+\;& D_3 <\theta u_p> b_{kj} + D_4(<\theta u_k> b_{pj} + <\theta u_j> b_{pk}) \\
+\;& D_5 <\theta u_p> b^2_{kj} + D_6(<\theta u_k> b^2_{pj} + <\theta u_j> b^2_{pk}) \\
\\
+\;& [D_7 b_{qp}\delta_{kj} + D_8(b_{qk}\delta_{pj} + b_{qj}\delta_{pk})] <\theta u_q> \\
+\;& [D_9 b_{qp}b_{kj} + D_{10}(b_{qk}b_{pj} + b_{qj}b_{pk})] <\theta u_q> \\
+\;& [D_{11} b_{qp}b^2_{kj} + D_{12}(b_{qk}b^2_{pj} + b_{qj}b^2_{pk})] <\theta u_q> \\
\\
+\;& [D_{13} b^2_{qp}\delta_{kj} + D_{14}(b^2_{qk}\delta_{pj} + b^2_{qj}\delta_{pk})] <\theta u_q> \\
+\;& [D_{15} b^2_{qp}b_{kj} + D_{16}(b^2_{qk}b_{pj} + b^2_{qj}b_{pk})] <\theta u_q> \\
+\;& [D_{17} b^2_{qp}b^2_{kj} + D_{18}(b^2_{qk}b^2_{pj} + b^2_{qj}b^2_{pk})] <\theta u_q>
\end{aligned}
$$

where b_{ij} is the anisotropy tensor $b_{ij} = <u_i u_j> / <u_q u_q> - 1/3\delta_{ij}$ and $<\theta u_i>$ is the turbulent heat-flux. Following Pope's linearity principle only terms linear in the heat-flux are kept. The A_i and D_i are functions of the invariants of b_{ij} and $<\theta u_i>$.

Parameterizing the integral of a two-point correlation in terms of the local anisotropy tensor is a substantial simplification requiring consideration. For a turbulence with short term memory and limited awareness Lumley (1970) has discussed the conditions under which such a constitutive relation is tenable and carried out a similar expansion procedure, Lumley (1967), indicating how the truncation errors scale. ¿From one point of view, the constitutive relation proposed can be seen as the first term in a functional Taylor series expansion for the rapid-pressure correlation. As the correlation decays with distance the primary contribution to the integral will come from regions within an integral length scale of the local position - thus, in a homogeneous turbulence, the first term will constitute a good approximation. Retaining higher order terms of the functional Taylor series expansion which involves spatial and temporal derivatives of b_{ij} substantially complicates the problem. It is expected that the retention of only the first order term captures enough of the physics to allow prediction suitable for engineering purposes.

In applying the above constitutive relation in non-inertial systems there are several phenomena, peculiar to rotating flows, that may be in conflict with the assumptions of the turbulence closure. In particular the inertial wave field associated with the rotation may interfere with: 1) the energy cascade from the large to the small scales of the flow, 2) the universal equilibrium assumed for the small scales of the flow, and 3) the assumed steadiness of the mean flow. These issues require consideration.

In a mean flow rotating with speed Ω there will be inertial oscillations with frequency less than or equal to 2Ω. For a quasi-steady assumption to be valid, changes in the mean flow must be slow with respect to the turbulence's ability to adjust to the imposed changes. This requires that $2\Omega < \varepsilon/k$ and becomes a lower bound for a Rossby number, $Ro = \varepsilon/2\Omega k > 1$, for which an equilibrium theory is appropriate.

Similar to a stably stratified density field in which stratification inhibits particle motions in the vertical direction, rotation inhibits transverse displacements of fluid particles. Similar to the radius of gyration of a charged particle in a magnetic field, the lateral displacement of a turbulent fluid element in a rotating frame can be characterized by a length scale $(q^2/3)^{1/2}/2\Omega$ where $q^2 =< u_j u_j >$. This length scale must be larger than the turbulence length scale $\ell \sim (q^2/3)^{3/2}/\varepsilon$ to insure that the transverse confinement by the Coriolis forces does not affect the fluctuating field. This produces a similar bound on the turbulent Rossby number $Ro = \varepsilon/2\Omega k > 3/2$.

The phase coherence necessary for the cascade of energy to the smaller scales of the motion will be interfered with if the production scales of the motion $\kappa\ell \sim 1$, where κ is the wavenumber, are subject to an inertial wave field. A different Rossby number can be defined as a ratio of the vorticity of the production scales of the motion to the background vorticity $Ro_t = (q^2/3)^{1/2}/2\Omega\ell = 3\varepsilon/2\Omega q^2$ using $\varepsilon = (q^2/3)^{1/2}/\ell$. A spectral Rossby number can also be defined as $Ro(\kappa) = u(\kappa)/2\Omega\ell(\kappa) = (\kappa E(\kappa))^{1/2}/2\Omega(2\pi/\kappa)$ which using the inertial range scaling $E(\kappa) = \alpha\varepsilon^{2/3}\kappa^{-5/3}$ and $\varepsilon = (q^2/3)^{1/2}/\ell$ becomes $Ro(\kappa) \sim 0.2(\kappa\ell)^{2/3}Ro_t$. The effects of rotation decrease as the wave number increases. For the inertial oscillations associated with the rotation not to interfere with the cascade mechanism $Ro(\kappa) > 1$ for $\kappa\ell \sim 1$ is required. Thus for $Ro_t \geq 5$ the usual parameterization of the spectral cascade rate, ε, in terms of the energy containing scales of the motion is appropriate. For $Ro_t < 5$ the current dissipation equation requires modification. How the dissipation equation is to be changed to account for the effects of rotation on the cascade rate is an unresolved issue that is the topic of current research. It is, however, clear that the assumption of the small scale equilibrium with the large scales of the motion is valid in most high Reynolds number rotating flows of interest: the Rossby number of the dissipation scales of the motion is $Ro_\varepsilon = Re_\ell^{1/2} Ro_t$ and therefore the dissipation scales are insensitive to the effects of rotation.

Having dealt with some of the phenomenological issues, the mathematical constraints required of the pressure-velocity correlation are now investigated.

There are several physical inequalities and mathematical identities the **X** must satisfy. These principles are used to obtain a set of algebraic constraint equations for the A_i and D_i. For an arbitrary three-dimensional turbulence the tensor polynomials must satisfy the symmetry constraints,

$$\begin{aligned} X_{ijkl} &= X_{ijlk} \\ X_{ijkl} &= X_{jikl} \\ X_{ijk} &= X_{ikj}. \end{aligned}$$

These symmetry constraints are built in to the assumed form of the tensor polynomials. For an arbitrary three-dimensional turbulence the tensor polynomials must also satisfy the constraints of normalization, continuity:

$$\begin{aligned} X_{ijkk} &= < u_i u_j > & X_{ikk} &= < u_i \theta > \\ X_{ijjk} &= 0 & X_{iji} &= 0. \end{aligned}$$

Note that a contraction of the integral of a two-point statistic is a local one-point statistic.

The tensors $< u_i u_j >$ and $< \theta\theta >< u_i u_j > - < \theta u_i >< \theta u_j >$ are positive semi-definite. This reflects the fact that the energy of the turbulence is always positive and that the magnitude of the correlation coefficients between the various components of the tensors be bounded by one. These facts lead to the "realizability" and "joint-realizability" constraints which specify the behavior of the correlations when specific limit states are approached. The relevant portion of the Reynolds stress transport equations, in principal axes, requires that

$$D/Dt \; < u_\alpha u_\alpha > \sim \; [U_{p,i} + \epsilon_{ipk}\Omega_k Ro^{-1}] X_{i\alpha p\alpha} \rightarrow 0 \quad \text{as} \qquad < u_\alpha u_\alpha > \rightarrow 0$$

in order to satisfy realizability. The rate of change, due to the rapid-pressure correlation, of the eigenvalue $< u_\alpha u_\alpha >$ is required to vanish as the limit state is approached. This insures that the rapid-pressure correlation model does not cause the solution to go into the unrealizable region in which $< u_\alpha u_\alpha >$ is negative. This realizability limit is rephrased in terms of the determinant of the Reynolds stress: $F = (R_{jj}^3 - 3R_{jj}R_{jj}^2 + 2R_{jj}^3)/6$ where $R_{ij} =< u_i u_j > / < u_p u_p >$ which can be written in terms of the invariants of the anisotropy tensor as $F = 1 + 9II + 27III$ where $II = -1/2 b_{ij} b_{ij} = -1/2 < b^2 >$, $III = 1/3 b_{ip} b_{pj} b_{ji} = 1/3 < b^3 >$. The determinant F varies between zero and one; $F = 1$ corresponds to an isotropic turbulence and $F = 0$ corresponds to the realizable limit.

Similar reasoning applied to the mixed tensor involving the Reynolds stress, the heat flux and the variance of the temperature fluctuations, produces the "joint-realizability" constraint

$$D/Dt\ D_{\alpha\alpha} \sim [U_{p},_{i} + \epsilon_{ipk}\Omega_k Ro^{-1}][< \theta\theta > X_{i\alpha p\alpha} - < \theta u_\alpha > X_{i\alpha p}] \to 0 \quad \text{as} \quad D_{\alpha\alpha} \to 0$$

which couples the rapid-pressure correlations appearing in the heat flux and the Reynolds stress equations. A similar determinant function F_d is defined with the normalized D_{ij}, for which $0 \leq F_d \leq 1$. Joint-realizability reflects the requirement that the magnitude of the correlation coefficients be bounded by one: the Reynolds stress and the heat-flux take on values "jointly" such that the time rate of change of $D_{\alpha\alpha}$ goes to zero as $D_{\alpha\alpha}$ goes to zero. Additional detail regarding the application of realizability constraints can be found in Shih and Lumley (1985).

Note that no assumptions regarding the higher order derivatives of the eigenvalue have been made. The strong form of realizability, in which $D^2/Dt^2 \ < u_\alpha u_\alpha > > 0$ is required at the realizability limit in order to allow the turbulence to leave the realizable state is not invoked. Such an agency is already present in the slow terms and it is not necessary to force the rapid terms to be responsible for such behavior which, as will be seen, is inconsistent with the a small parameter expansion of the rapid-pressure representation around the realizability limit. Instead a weak realizability constraint, as specified by Speciale *et al.* (1993), Pope (1983), which does not allow the solution to attain the realizable limit in finite time is invoked. This is done by requiring that the rapid-pressure correlation vanish more rapidly than the slow-pressure correlation model. This avoids any assumptions regarding the behavior of the second derivative which are required for the flow to leave the realizable state which is accessible in finite time in models using the strong form of the realizability constraint. Moreover recent work by Speziale *et al.* (1993) indicates that the present hierarchy of second-order models is inconsistent with the strong form of the realizability constraint. The rate of rotation of the eigenvalues coming from the second derivative is a sink term of a form that that cannot be balanced by the present models. In setting the portion of $D/Dt \ < u_\alpha u_\alpha >$ due to the rapid-pressure correlation to zero while choosing a return term model that precludes accessibility of the realizable limit state the weak form of realizability is satisfied by the sum of the modeled terms on the right hand side of the transport equations. The issue raised by Speziale *et al.* (1993) does not impact on the present model, as all the higher order derivatives vanish, and the realizable limit is not attainable in finite time because the relative rate of dissappearance of the slow-pressure terms with respect to the rapid-pressure terms. It may, however, require a rethinking of the modeling principles so that the intrinsic negativity of the second derivative can be properly balanced if the realizable limit is to be considered accessible.

The last of the constraints to be applied to specify the constitutive relation, the geostrophic constraint, is now discussed. The derivation of the constraint, as it has not been given elsewhere in the published literature, Ristorcelli (1991), is given. For a turbulence two-dimensional in planes perpendicular to the axis of rotation Hide(1977) and Speziale (1985, 1990) have shown that the equations are materially-frame-indifferent. This is a direct consequence of the Taylor-Proudman theorem. For a modeled set of equations to be consistent with the 2DMFI principle the rapid-pressure correlations must satisfy the "geostrophic" constraints,

$$\begin{aligned} \epsilon_{pqn}\Omega_n X_{ipqj} = &\quad \epsilon_{qjn}\Omega_n < u_i u_q > \\ \epsilon_{pqn}\Omega_n X_{pqi} = &\quad \epsilon_{qin}\Omega_n < u_q \theta > . \end{aligned}$$

Consider the portion of the rapid-pressure correlation associated with the rotation

$$\epsilon_{pqk}\Omega_k Ro^{-1} X_{ipqj} = \epsilon_{pqk}\Omega_k (4\pi Ro)^{-1} \int < u_i(\mathbf{x}) u_p(\mathbf{x}') >,_{j'q'} d\mathbf{x}'/(\mathbf{x}-\mathbf{x}').$$

A velocity field, two-dimensional in planes perpendicular to the axis of rotation, has the representation $u_p = \epsilon_{pqk}\Omega_k \psi,_q$. Inserting the expression for the velocity field into the integral and contracting produces, in the integrand, the Laplacian of the streamfunction, $\psi,_p = \epsilon_{qpk}\Omega_k u_q$ which reduces the volume integral of a two-point statistic to a local one-point statistic,

$$\epsilon_{pqk}\Omega_k Ro^{-1} X_{ipqj} = Ro^{-1} < u_i \psi,_j >= Ro^{-1}\epsilon_{pjk}\Omega_k < u_i u_p >$$

upon application of Green's theorem. The geostrophic constraint $\epsilon_{pqn}\Omega_n X_{ipqj} = \epsilon_{qjn}\Omega_n < u_i u_q >$ then follows. The satisfaction of the geostrophic constraint means that, when the turbulence is two-dimensional in planes perpendicular to the axis of rotation, the portion of the rapid-pressure correlation associated with the rotation is equal to the Coriolis terms effectively removing any dependence on the rotation rate from the equations. Such a Coriolis-pressure force balance is found in the large scale atmosphere were it describes the well-known geostrophic wind. A point on nomenclature: a geostrophic turbulence is one that is two-dimensional because its Rossby number is small. This latter qualification distinguishes it from a two-dimensional turbulence (that is made so by some other means) for arbitrary Rossby number. Both, however, are frame-indifferent.

The application of the five sets of constraints - normalization, continuity, realizability, joint-realizability and geostrophy - produce thirty-six linear algebraic equations (several of which are redundant) for the thirty-three unknown coefficients $A_i, (i = 1, 15)$ and $D_i, (i = 1, 18)$ appearing in the tensor polynomials. The equations are of the general form $A_{ij}(II, III)x_j = b_i$ where $x_i = [A_1, ..., A_{15}, D_1, ..., D_{18}]$ and where II and III are the invariants of the anisotropy tensor, $II = -1/2 b_{ij} b_{ij} = -1/2 < b^2 >$, $III = 1/3 b_{ip} b_{pj} b_{ji} = 1/3 < b^3 >$. Using the definition $F = 1 + 9II + 27III$ the general form of the equations can be rewritten as $A_{ij}(II, F)x_j = b_i$. Note that F and II appear linearly in the constraint equations.

The *ansatz*

$$\begin{aligned} X_{ijkl} = &\quad X^0_{ijkl} + F X^F_{ijkl} \\ X_{ijk} = &\quad X^0_{ijk} + F X^F_{ijk} \end{aligned}$$

is used to extract more information from the constraint equations. Here $\mathbf{X}^0$ satisfies the set of constraint equations $A_{ij}(II, 0)x^0_j = b_i$ obtained by application of all five sets of constraints which are simultaneously valid only when $F = 0$ (or equivalently when $III = -(II + 1/9)/3$). While $\mathbf{X}^F$ is obtained from the reduced set of equations $A_{ij}(II, F)x_j^F = b_i^0$ which satisfy the *three* sets of constraints of normalization, continuity and joint-realizability and where b_i^0 is a known function of the $\mathbf{X}^0$ solution. As there are more unknowns than equations there are free parameters which will be used shortly to insure that the model is asymptotically consistent with an equilibrium state. The solution, in which all free parameters are set to zero, and which satisfy all the mathematical constraints, will be called the basic model and has been given in Ristorcelli (1987,1991) and Ristorcelli and Lumley (1991b). These coefficients are given in Appendix 3. It is to this basic model that additional terms are added in order to insure that the model is asymptotically consistent with a known equilibrium state of a particular turbulent flow field.

In reference to the coefficients, A_i, of the basic model and defined in Appendix 3, several comments are appropriate. The A_i are nonlinear functions of the invariants of b_{ij}: they are ratios of polynomials of the invariants. The realizability limit is attained along the line $III = -(II + 1/9)/3$ on which $F = 0$. At the isotropic limit $II = III = 0$ and $F = 1$, and thus $A_1 + A_2 = 1/10$ and the well known exact result for isotropic turbulence is obtained. Note that the "off-realizability" correction $F\mathbf{X}^F$ is necessary to obtain this limit. The satisfaction of the isotropic limit is discovered to be a *consequence* of the constraints used to create the 2DMFI model - it is not a constraint that has been enforced to obtain the model but is satisfied naturally by the *ansatz*. This seems to vindicate the present procedure.

Note that the coefficients appear to be singular at the one-dimensional limit when $1 + 3II = 0$. This is actually not the case as the singularities arising from the individual terms in the rapid pressure representation annihilate each other when summed. This has been verified analytically near the one dimensional limit, as $II \to -1/3$, in principle axes of the Reynolds stress tensor using a perturbation expansion.

It should be emphasized that the form of the rapid-pressure correlations was chosen so that the coefficients *are valid for all states of the turbulence* - they are not fixed to their values at the realizable limit. The resulting model therefore satisfies all mathematical constraints for any arbitrary Reynolds stress - *not just at the realizability or geostrophic limit states.* The second term, $F\mathbf{X}^F$, in the expression for $\mathbf{X}$, represents the "off-realizability correction", thus the coefficients, A_i and D_i, are functions dependent on the state of the turbulence as parameterized by the invariants and only attain their realizable limit values at $F = 0$.

4. A compact representation of the 2DMFI rapid-pressure model

The rapid-pressure correlation models are usually written in the form they appear in the Reynolds stress equations with:

$$\begin{aligned}\prod_i^r &= 2[U_{q},_p + \epsilon_{pqk}\Omega_k Ro^{-1}]X_{piq} \\ \prod_{ij}^r &= 2[U_{q},_p + \epsilon_{pqk}\Omega_k Ro^{-1}][X_{ipqj} + X_{jpqi}].\end{aligned}$$

which can be rewritten in terms of the strain and rotation tensors as

$$\prod_{ij}^r = 2[S_{qp} + W_{qp}][X_{ipqj} + X_{jpqi}]$$

where $S_{qp} = 1/2(U_{q,p} + U_{p,q})$ and $W_{qp} = [1/2(U_{q,p} - U_{p,q}) + \epsilon_{pqk}\Omega_k Ro^{-1}]$ are the usual mean strain rate and the total or intrinsic rotation rate. The second-order equations are then rewritten as

$$D/Dt < u_i u_j > + 2\epsilon_{ikp} < u_p u_j > \Omega Ro^{-1} + 2\epsilon_{jkp} < u_p u_i > \Omega_k Ro^{-1} = + \prod_{ij}^r + \prod_{ij}^s + \ldots$$

$$D/Dt < \theta u_i > + 2\epsilon_{pik}\Omega_k < \theta u_p > Ro^{-1} = + \prod_i^r + \prod_i^s + \ldots$$

where the terms omitted have already been given. Taking the contraction of the fourth-order tensor on the mean velocity gradients to obtain the form used in the Reynolds stress equations produces, with $q^2 =< u_p u_p >$,

$$\begin{aligned}\prod_{ij}^r/2q^2 = &\ 2[A_1 + A_2]S_{ij} \\ + &\ [(A_3 + A_4 + 2A_5)(b_{ip}S_{pj} + b_{jp}S_{pi}) + 2A_5\delta_{ij} < bS >] \\ + &\ [(A_3 - A_4)(b_{ip}W_{pj} + b_{jp}W_{pi})] \\ + &\ [(A_6 + A_7 + 2A_8)(b^2_{ip}S_{pj} + b^2_{jp}S_{pi}) + 2A_8\delta_{ij} < b^2S >] \\ + &\ [(A_6 - A_7)(b^2_{ip}W_{pj} + b^2_{jp}W_{pi})] \\ + &\ 2[(A_9 + A_{10})b_{ip}S_{pq}b_{qj} + A_{10}b_{ij} < bS >] \\ + &\ [(A_{11} + A_{12} + 2A_{13})(b_{ip}S_{pq}b^2_{qj} + b_{jp}S_{pq}b^2_{qi}) + 2A_{13}(b^2_{ij} < bS > + b_{ij} < b^2S >)] \\ + &\ [(A_{11} - A_{12})(b_{ip}W_{qp}b^2_{qj} + b_{jp}W_{qp}b^2_{qi})]\end{aligned}$$

Here the angle brackets represent the trace of the indicated quantity: *eg.* $< bS >= b_{ij}S_{ij}$ and that $II = -1/2 < b^2 >$, $III = 1/3 < b^3 >$. Note that the $\prod_{ij}^r$ has zero trace because of the continuity constraint, $X_{ijjk} = 0$, requires $A_3 + A_4 + 5A_5 - II(A_{11} + A_{12} + 4A_{13}) = 0$ and $A_6 + A_7 + 5A_8 + A_9 + A_{10} = 0$ It is possible to rewrite the higher order tensor bases in terms of the lower order terms substantially simplifying the form of the model. The generalized Cayley-Hamilton theorem is used to rewrite the expression in an irreducible tensor basis. Using the matrix notation

$$\begin{aligned}\mathbf{bSb} &= -[\mathbf{b}^2\mathbf{S} + \mathbf{Sb}^2] + < \mathbf{bS} > \mathbf{b} + 1/2 < \mathbf{b}^2 > \mathbf{S} + < \mathbf{b}^2\mathbf{S} > \mathbf{1} \\ \mathbf{bSb}^2 + \mathbf{b}^2\mathbf{Sb} &= -1/3 < \mathbf{b}^3 > \mathbf{S} + < \mathbf{b}^2\mathbf{S} > \mathbf{b} + < \mathbf{bS} > \mathbf{b}^2\end{aligned}$$

the 2DMFI rapid-pressure correlation can be written more compactly as

$$\begin{aligned}\prod_{ij}^r/2q^2 = &\ [B_3 + < b^2 > B_3'' + < b^3 > B_3''']S_{ij} \\ + &\ B_4[b_{ip}S_{pj} + b_{jp}S_{pi} - 2/3 < bS > \delta_{ij}] + B_4''' < bS > [b^2_{ij} + 2II/3\delta_{ij}] \\ + &\ B_5[b_{ip}W_{pj} + b_{jp}W_{pi}] + [B_6 < bS > + B_4''' < b^2S >]b_{ij} \\ + &\ B_7[b^2_{ip}S_{pj} + b^2_{jp}S_{pi} - 2/3 < b^2S > \delta_{ij}] \\ + &\ B_8[b^2_{ip}W_{pj} + b^2_{jp}W_{pi}] + B_9[b_{ip}W_{qp}b^2_{qj} + b_{jp}W_{qp}b^2_{qi}]\end{aligned}$$

where the values of the B_i in terms of the A_i are given in Appendix 1. The above representation of the rapid-pressure correlation will be called the basic or uncalibrated model. The coefficients, B_i, appearing in the base 2DMFI model above come from first principles: they do not result from any numerical optimization with experimental or numerical data. Comparisons to the FLT model, Fu *et al.* (1987), shows that the two models have the same tensor structure containing the same generators. The only difference is in the terms proportional to $[b_{ij}]$ and $[b^2_{ij} + 2II/3\delta_{ij}]$, usually identified with the slow-pressure's contribution to the pressure-strain correlation and calibrated accordingly, now reflect a contribution from the rapid-pressure. Contraction of the irreducible form of X_{ijkl} on the mean velocity gradients has produced the bases $[b_{ij}]$ and $[b^2_{ij} + 2II/3\delta_{ij}]$ in which the coefficients are functions of the invariants $< bS >$ and $< b^2S >$ in addition to the dependence on $< b^2 >$ and $< b^3 >$ appearing in the A_i. The two tensor bases, usually associated with the slow-pressure correlation, arise as a consequence of starting with the two-point volume integral and represent a contribution to the pressure-strain correlation whose structure is identical to the slow-pressure models but whose genesis is in the rapid-pressure.

Speziale *et al.* (1991) have also written a general form for the pressure-strain covariance. It is linear in the mean velocity gradients and nonlinear in the anisotropy tensor. Their expression contains the same generators as the present model except for the cubic term $[\mathbf{bWb}^2 - \mathbf{b}^2\mathbf{Wb}]$. Speziale *et al.* (1991) use results from rational mechanics (*cf.* Smith (1971)) to expand in a functional basis. The present strategy uses a polynomial basis for which the generator $[\mathbf{bWb}^2 - \mathbf{b}^2\mathbf{Wb}]$ is not redundant, Spencer (1971). The tensor polynomial given above is irreducible and the basis is optimal. Speziale *et al.* (1991) have shown, for planar flows, that the generators nonlinear in the anisotropy tensor can be expressed in terms of generators linear in the anisotropy tensor. This fact led to the very simple form of the SSG model, Speziale *et al.* (1991). This fact, from a rigorous though not necessarily practical point of view, limits the model to planar flows. The results of Speziale *et al.* (1991) can be used to recast the present model into its *linear* planar form. For planar flows Speziale *et al.* (1991) has shown that

$$[b^2_{ip}S_{pj} + b^2_{jp}S_{pi} - 2/3 < b^2S > \delta_{ij}] = -b_{33}[b_{ip}S_{pj} + b_{jp}S_{pi} - 2/3 < bS > \delta_{ij}] - 2/3(III/b_{33})S_{ij}$$

$$[b^2_{ip}W_{pj} + b^2_{jp}W_{pi}] = -b_{33}[b_{ip}W_{pj} + b_{jp}W_{pi}]$$

from which it follows that

$$[b_{ip}W_{qp}b^2_{qj} + b_{jp}W_{qp}b^2_{qi}] = (II + b_{33}b_{33})[b_{ip}W_{pj} + b_{jp}W_{pi}]$$

and the planar form of the 2DMFI rapid-pressure model can be written as

$$\begin{aligned} \prod^r_{ij}/2q^2 = \; & (C_3 - 2IIC''_3 + 3IIIC'''_3 - 2/3(III/b_{33})C_7)[S_{ij}] \\ + \; & (C_4 - b_{33}C_7)[b_{ip}S_{pj} + b_{jp}S_{pi} - 2/3 < bS > \delta_{ij}] + C'''_4 < bS > [b^2_{ij} + 2II/3\delta_{ij}] \\ + \; & (C_5 - b_{33}C_8 + (II + b_{33}b_{33})C_9)[b_{ip}W_{pj} + b_{jp}W_{pi}] + (C_6 < bS > + C'''_4 < b^2S >)[b_{ij}] \end{aligned}$$

which has the same linear tensor bases as the quasi-linear SSG model. The form of the model in planar flows is consistent with to the SSG model which is the topologically generic form of a general class of models for the fourth order rapid-pressure tensor X_{ijkl}. The coefficients in the model, however, are nonlinear functions of the invariants while in the SSG model, given below for comparison,

$$\begin{aligned} \prod_{ij}^{r} = & -(2C_1\varepsilon + C_1^*\mathcal{P})[b_{ij}] + C_2\varepsilon[b_{ij}^2 + 2II/3\delta_{ij}] + (C_3 - (-2II)^{1/2}C_3^*)k[S_{ij}] \\ + & \; C_4 k[b_{ip}S_{pj} + b_{jp}S_{pi} - 2/3 < bS > \delta_{ij}] + C_5 k[b_{ip}W_{pj} + b_{jp}W_{pi}] \end{aligned}$$

the C_i are numerical constants and one coefficient in the $[S_{ij}]$ term, in the rapid-pressure portion of the model is nonlinear. The C_i in the SSG model are determined by a numerical optimization so that the model reproduces as closely as possible: 1) the stationary state of the homogeneous shear and 2) maximizes the kinetic energy growth rate of the rotating homogeneous shear as close as possible to the $\Omega/S = 0.25$ predicted by rapid distortion theory without introducing a Richardson number similarity, Speziale and Mhuiris (1989), while insuring that the points of exchange of stability are outside of those predicted by the linear theory of Bertoglio (1982).

In the return term of the SSG model the correction to the linear b_{ij} term arrived at more or less intuitively by Speziale *et al.* (1991) is, to lowest order, vindicated by the present results. Speziale *et al.* (1991) have altered the term linear in b_{ij}, usually associated with the slow-pressure correlation, to include a term involving the mean flow, a term proportional to the production $\mathcal{P} = -2 < u_i u_j > U_{i,j}$: the usual $C_1\varepsilon b_{ij} \rightarrow (C_1\varepsilon + C_1^*\mathcal{P})b_{ij}$. The present analysis indicates that the portion of the rapid-pressure contribution to the pressure-strain correlation, linear in b_{ij}, has the form $[B_6 < bS > + B_4''' < b^2 S >]b_{ij}$. Note that $< bS >$ can be written in terms of the production as $< bS >= -\mathcal{P}/q^2$. The present analysis suggests the possibility of adjusting the nonlinear return term for mean velocity gradient effects in a similar way. In the present model the portion of the rapid-pressure quadratic in the anisotropy tensor is $B_4''' < bS > [b_{ij}^2 + 2II/3\delta_{ij}]$ and therefore also scales with the production.

Speziale *et al.* (1992) have reflected on the ambiguity of the distinction between the rapid- and slow-pressure contributions to the pressure-strain correlation. They have modeled the whole pressure-strain correlation but whether to interpret their adjustments (as described in the previous paragraph) as incorporating the effects of the mean strain on the return terms or as contributions of the rapid-pressure to the total pressure-strain is not clear. Here, however, the distinction is clear. The present analysis starts from the tensor polynomial representation for the rapid-pressure integral and produces terms whose tensor structure is identical to that which are traditionally called the slow-pressure. The other non-linear pressure-strain models, FLT or the SL model, Shih and Lumley (1985), do not have terms in b_{ij} or b_{ij}^2 that can be similarly identified.

The derivation of a rapid-pressure correlation satisfying all the limit states and valid away from the limit state, though mathematically rigorous, will not necessarily produce a model that performs better than other models in flows for which latter models are calibrated. However, for the general class of flows for which second-order models are suitable we are of the opinion that the satisfaction of all mathematical constraints constitutes a necessary (though not sufficient) requirement to assure the predictive capabilities in flows different from those for which the models are calibrated. The existence of several free parameters can then be used to calibrate the model to specific classes of flows of computational interest. This tack is taken in the next section where the fixed points of the modeled equation are matched to the fixed points of the homogeneous shear. In this way a model for the class of flows in which the mean shear is the dominant production mechanism is created.

5. Calibrating the rapid-pressure correlation representation

The coefficients, B_i, appearing in the 2DMFI model come from first principles: they do not result from any calibration with experimental or numerical data. They represent the *minimum* number of determined coefficients necessary to satisfy the mathematical constraints on the rapid-pressure correlation for a three-dimensional Reynolds stress. It, however, is not a unique representation: there are an infinity of solutions corresponding to different values of the free parameters which, in the basic model representation shown, have been set to zero. Computations have shown that the predictive capabilities of the basic model are inadequate. The mathematics built into the model do not capture the experimentally known stationary points of homogeneous shear. To compensate for some of the approximations made in the mathematical development the model is modified so that the modeled evolution equations have the same asymptotic behavior as that observed: to the basic model additional terms are added to insure that the model is consistent with an equilibrium state of a particular benchmark turbulent field. This is done without sacrificing any of the mathematical principles built into the model. It, however, can not be done arbitrarily.

The strategy is to require asymptotic consistency with an equilibrium state. The modeled evolution equations are required to have the same fixed points as those observed in experiment. The equilibrium constraint is used to obtain additional constraints equations to specify the free parameters. This is very similar to the strategy employed so far in that limit states are used to set model coefficients. Here, of course, the equilibrium state is much closer to those expected to be seen in flows of engineering interest. The free parameters will now be called calibration coefficients, A_i^c, and will be collectively denoted by X_{ijkl}^{∞} appearing in the decomposition

$$X_{ijkl} = \quad X_{ijkl}^{0} + FX_{ijkl}^{F} + FX_{ijkl}^{\infty}.$$

X_{ijkl}^{∞} represents the additional constant terms necessary to capture the stationary state. This form is equivalent to assuming that the coefficients in the tensor polynomial have the form $A_i = A_i^0 + FA_i^F + FA_i^c$. The A_i^c satisfy normalization and continuity constraints and the algebraic equations describing the stationary state of a homogeneous shear. It is at this point that numerical or experimental data for the asymptotic values of the anisotropy tensor, the production to dissipation, $(\mathcal{P}/\varepsilon)_\infty$, and the ratio of time scales, $(Sk/\varepsilon)_\infty$, are necessary. The details of this strategy are now given.

The A_i^c satisfy the six equations given by the homogeneous form of normalization and continuity constraints, $X^\infty_{ijkk} = 0, X^\infty_{ijjk} = 0$, which are valid for all states of the turbulence. Substituting in the calibration coefficients allows six A_i^c, to be expressed in terms of seven free parameters,

$$\begin{aligned}
A_1^c =& \ -(50A_8^c + 12A_9^c + 4A_{10}^c)II/15 + (A_{11}^c + A_{12}^c - 6A_{13})III/5 \\
A_2^c =& \ (20A_8^c + 3A_9^c + A_{10}^c)II/15 - (3A_{11}^c + 3A_{12}^c + 2A_{13})III/10 \\
A_3^c =& \ -11/3A_5^c + (A_{11}^c + 3A_{12}^c + 8A_{13}^c)II/3 \\
A_4^c =& \ -4/3A_5^c + 2(A_{11}^c + 2A_{13}^c)II/3 \\
A_6^c =& \ -1/3(11A_8^c + 3A_9^c + A_{10}^c) \\
A_7^c =& \ -2/3(2A_8^c + A_{10}^c).
\end{aligned}$$

The seven parameters will be determined by asymptotic consistency with a particular equilibrium state. At this point the rapid-pressure correlation model is fully general and it is possible to write it in its final form, without specifying the calibration coefficients, as

$$\begin{aligned}
\prod\nolimits^r_{ij}/2q^2 =& \ [C_3 - 2IIC_3'' + 3IIIC_3''']S_{ij} \\
+& \ C_4[b_{ip}S_{pj} + b_{jp}S_{pi} - 2/3 < bS > \delta_{ij}] + C_4''' < bS > [b^2_{ij} + 2II/3\delta_{ij}] \\
+& \ C_5[b_{ip}W_{pj} + b_{jp}W_{pi}] + [C_6 < bS > + C_4''' < b^2S >]b_{ij} \\
+& \ C_7[b^2_{ip}S_{pj} + b^2_{jp}S_{pi} - 2/3 < b^2S > \delta_{ij}] \\
+& \ C_8[b^2_{ip}W_{pj} + b^2_{jp}W_{pi}] + C_9[b_{ip}W_{qp}b^2_{qj} + b_{jp}W_{qp}b^2_{qi}].
\end{aligned}$$

The calibration coefficients, A_i^c, have been added to the base model coefficients, B_i, to obtain the final model coefficients C_i:

$$\begin{aligned}
C_3 =& \ B_3 - 2F(10A_8^c + 3A_9^c + A_{10}^c)II/5 - F(A_{11}^c + A_{12}^c + 14A_{13}^c))III/5 \\
C_3'' =& \ B_3'' + F(A_9^c + A_{10}^c) \\
C_3''' =& \ B_3''' - 1/3F(A_{11}^c + A_{12}^c + 2A_{13}^c) \\
C_4 =& \ B_4 + F(-3A_5^c + II(A_{11}^c + A_{12}^c + 4A_{13}^c)) \\
C_4''' =& \ B_4''' + F(A_{11}^c + A_{12}^c + 4A_{13}^c) \\
C_5 =& \ B_5 + F(-7/3A_5^c + (-A_{11}^c + 3A_{12}^c + 4A_{13}^c)II/3) \\
C_6 =& \ B_6 + F(2A_9^c + 4A_{10}^c) \\
C_7 =& \ B_7 - 3F(A_8^c + A_9^c + A_{10}^c) \\
C_8 =& \ B_8 - 1/3F(7A_8^c + 3A_9^c - A_{10}^c) \\
C_9 =& \ B_9 + F(A_{11}^c - A_{12}^c)
\end{aligned}$$

Note that the traces can be written in terms of the production $< bS >= -\mathcal{P}/q^2$ and $< b^2S >= 1/2b_{33}\mathcal{P}/q^2$; in planar flow $< bS >= -2b_{12}S$ and $< b^2S >= -b_{12}b_{33}\mathcal{P}/q^2$.

The calibration coefficient A_5^c involves adjustments to the generators $[\mathbf{bS} + \mathbf{Sb} - 2/3 < bS > \mathbf{1}]$ and $[\mathbf{bW} - \mathbf{Wb}]$ which are linear in the anisotropy tensor. Numerical experiments have indicated that it is important in establishing the levels of the normal components of the Reynolds stresses. Combinations of A_8^c, A_9^c and A_{10}^c affect the generators quadratic in the anisotropy tensor, $[\mathbf{b}^2\mathbf{S} + \mathbf{b}^2\mathbf{S} - 2/3 < b^2S > \mathbf{1}]$ and $[\mathbf{b}^2\mathbf{W} - \mathbf{W}\mathbf{b}^2]$ and to a very small degree the $\mathbf{S}$ term. Combinations of only A_9^c and A_{10}^c can be used to control the contribution to $\prod^r$ proportional to $\mathbf{b}$ and $\mathbf{S}$. Experience with the FLT model, Fu *et al.* (1987) indicates that the cubic term $[\mathbf{bWb}^2 - \mathbf{b}^2\mathbf{Wb}]$ involving the rotation tensor is important in controlling the relative level of the normal components of the Reynolds stress in situations with rotation. This suggests that the calibration coefficients A_{11}^c, A_{12}^c and A_{13}^c are important. The combination of calibration coefficients $A_{11}^c - A_{12}^c$ controls tensor products like $[\mathbf{bWb}^2 - \mathbf{b}^2\mathbf{Wb}]$ and combinations of terms A_{11}^c, A_{12}^c and A_{13}^c control the contributions of the $\mathbf{b}^2$ and $\mathbf{S}$.

The existence of the free parameters allows the model to be calibrated to a specific class of flows of computational interest. For example, in a buoyant flow one might evaluate the fixed points using the experiments of the homogeneous shear in a constant mean temperature gradient thus producing a model suitable for a class of stratified flows of geophysical interest. This calibration would, of course, involve the model for the rapid-pressure correlation appearing in the heat flux equation. For many engineering problems the primary production mechanism is the mean shear and capturing the fixed points of the homogeneous shear in the modeled equations will make the model suitable for a wide class of flows. At this point one could also consider using the exact results of rapid distortion theory to obtain values for the calibration coefficients. This, however, is inconsistent with the equilibrium hypothesis underlying the local approximation to the constitutive relation invoked for the rapid-pressure correlation integral. Moreover the rapid distortion problem is linear admitting a superposability that is not possible in the context of nonlinear second-order closures. The homogeneous shear structural equilibrium will be used to fix the representation for the rapid-pressure correlation.

The fixed points of the homogeneous shear are now built into the model by specifying the calibration coefficients. The modeled evolution equations for the homogeneous turbulence in a mean velocity gradient, are

$$\begin{aligned} D/Dt < u_i u_j >= \; & -2\epsilon_{ikp} < u_p u_j > \Omega_k - 2\epsilon_{jkp} < u_p u_i > \Omega_k - < u_i u_p > U_{j},_p - < u_j u_p > U_{i},_p \\ & + \textstyle\prod_{ij}^r - C_1 \varepsilon b_{ij} + C_2 \varepsilon [b_{ij}^2 + 2II\delta_{ij}/3] - 2/3\varepsilon\delta_{ij} \end{aligned}$$

$$D/Dt\, \varepsilon = -(C_{\varepsilon 1} < u_i u_j > U_{i},_j + C_{\varepsilon 2}\varepsilon)\, \varepsilon/k$$

assuming local isotropy for the dissipation. The terms $-C_1\varepsilon b_{ij} + C_2\varepsilon[b_{ij}^2 + 2II/3\delta_{ij}]$ represent the return to isotropy pressure correlation. Using $< u_i u_j >= q^2(b_{ij} + 1/3\delta_{ij})$ the equations for the anisotropy are

$$\begin{aligned} D/Dt\, b_{ij} = \; & -2\epsilon_{ikp} b_{pj}\Omega_k - 2\epsilon_{jkp} b_{pi}\Omega_k - \; [b_{ip}U_{j},_p + b_{jp}U_{i},_p - 2/3\delta_{ij} < bS >] \\ & -2/3S_{ij} + 2b_{ij} < bS > + \textstyle\prod_{ij}^r/q^2 - \; (C_1 - 2)b_{ij}\varepsilon/q^2 + C_2[b_{ij}^2 + 2II/3\delta_{ij}]\varepsilon/q^2. \end{aligned}$$

The mean strain and rotation tensors are $S_{ij} = 1/2S^*(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1})$ and $W_{ij} = 1/2W^*(\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1})$, where, $S = U_{1},_2 = S^* = W^*$. Setting the $D/Dt\, b_{ij} = 0$ produces three algebraic equations.

Inserting the experimentally determined asymptotic values of b_{ij} and Sk/ε produces three additional constraints leaving four of the seven A_i^c as free parameters. The data from the experiments of Tavoularis and Corrsin (1981), Champagne, Harris and Corrsin (1970), Tavoularis and Karnik (1989), and the DNS of Rogers *et al.* (1986) are summarized in the Table 1.

	TC	CHC	TK.A	TK.C	TK.D	TK.G	TK.J	TK.K	DNS
b_{11}^{∞}	0.197	0.137	0.217	0.257	0.197	0.157	0.157	0.147	0.215
b_{12}^{∞}	-0.14	-0.165	-0.165	-0.165	-0.17	-0.148	-0.154	-0.149	-0.158
b_{22}^{∞}	-0.143	-0.083	-0.133	-0.143	-0.133	-0.113	-0.103	-0.093	-0.153
b_{33}^{∞}	-0.053	-0.053	-0.083	-0.113	-0.063	-0.043	-0.053	-0.053	-0.062
$(P/\varepsilon)_\infty$	1.75	1.0	1.38	1.37	1.64	1.33	1.45	1.37	1.80
$(SK/\varepsilon)_\infty$	6.25	3.03	4.2	4.15	4.82	4.5	4.71	4.60	5.7

Table 1. Data for the homogeneous shear flow.

There is considerable scatter in the data due to the technique used to generate the turbulence, individual wind tunnels in which the different experiments were done, and experimental error. Not all the data, as has been qualified in the references from which the data is drawn, represent the asymptotic state. The following fixed point values are taken to be representative: $b_{11}^{\infty} = 0.203$, $b_{12}^{\infty} = -0.156$, $b_{22}^{\infty} = -0.143$, $b_{33}^{\infty} = -0.06$, $(Sk/\varepsilon)_\infty = 5.54$, $(P/\varepsilon)_\infty = 1.73$. They are obtained by a simple average of the data of TC, TK.D and the DNS. These three cases are chosen because they have the highest values of of the nondimensional time $(SK/\varepsilon)_\infty$ corresponding to the flows that are furthest in their development to the asymptotic state. The values of the invariants corresponding to these values of the anisotropy tensor are: $II_\infty = -0.057, III_\infty = 0.0032, F_\infty = 0.573$. Substituting these asymptotic values into the fixed point equations, $D/Dt\ b_{ij} = 0$, reduces the number of free parameters from seven to four.

$$\begin{aligned} A_5^c &= -0.29 - 0.06(A_{10}^c - A_8^c) \\ A_{11}^c &= -3.6 + 5A_{10}^c - 2A_{13}^c - 12.7A_8^c - 3.8A_9^c \\ A_{12}^c &= -24.5 - 44.2A_{10}^c - 2A_{13}^c + 29A_8^c - 8A_9^c \end{aligned}$$

This set of constraint equations is dependent on the model for the return to isotropy pressure. For simplicity the the nonlinear return coefficient has been set to zero, $C_2 = 0$, and the well accepted value, $C_1^{\infty} = 3.4$, has been chosen. There are still four free parameters. The following values of the four free parameters are chosen: $A_8^c = 0.8, A_9^c = -1.0, A_{10}^c = 0.01, A_{13}^c = 0$. The undetermined free parameters have been set by matching the values of the anisotropy for the log layer. The procedure is outlined in more detail in Appendix 1.

6. Computations and comparisons in homogeneous turbulence

The 2DMFI model falls into the same class of representations as the FLT and SL models: they all use nonlinear terms and invoke some form of realizability constraint to evaluate the coefficients. For this reason the 2DMFI model will be compared primarily to the nonlinear SL and FLT models. For completeness and because it appears to be a very successful model for the planar flows, computations with the quasi-linear SSG model are also shown. The SSG model is linear in the anisotropy tensor though nonlinear in that the scalar coefficients are functions of the invariants of the the anisotropy tensor. It should, however, be kept in mind that the SSG model satisfies realizability for the kinetic energy and not for the individual Reynolds stresses and is therefore in another class of models. This issue is more fully explored in a subsequent section. Results are not compared to the LRR model as, in concordance with the observations of Speziale *et al.* (1991), the SSG model is viewed as an updated optimized LRR model.

In all the calculations with the 2DMFI model a simple linear Rotta type model for the slow pressure correlation will be used. This corresponds to $C_2 = 0$ in the canonical form given above and is consistent with the present calibration to the homogeneous shear. For the linear return coefficient a simple expression, $C_1 = 2 - 31IIF^{1/2}$, is used. This satisfies the isotropic limit, $C_1 = 2.0$, and is consistent with the assumed value for the asymptotic homogeneous shear, $C_1^{\infty} = 3.4$. The form chosen is consistent with a weak form of realizability and the recent results of Speziale *et al.* (1993) regarding the rate of disappearance of the rapid-pressure correlation relative to the return pressure correlation as the realizability limit is approached.

The values used for the constants in the dissipation equation are: $C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.83$. Note that this corresponds to a single universal fixed point $(\mathcal{P}/\varepsilon)_{\infty} = 1.88$ independent of rotation. The single fixed point is a well-known deficiency common to all the present forms of the modeled dissipation equations.

Case 1: Homogeneous shear

The calibrated model is now used to compute the time evolution of the homogeneous shear flow. The mean strain and rotation tensors are $S_{ij} = 1/2S^{*}(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1})$ and $W_{ij} = 1/2W^{*}(\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1})$, where, $S = U_{1,2} = S^{*} = W^{*}$. In Figure 1 the time evolution of the turbulence energy is compared to the LES of Bardina *et al.* (1983), and the three models FLT, SL and SSG. A similar monotonic behavior is found for other statistics, b_{12}, b_{11}, II in the flow and, as they do not constitute new or different information, are not shown. In general, starting from physically realistic initial conditions, the flow attains its asymptotic state rapidly and monotonically. The asymptotic states which the different models attain are given in the accompanying table. The column labeled experimental data is an average of the three cases TC, DNS and TK.D.

Equilibrium Values	2DMFI Model	SL Model	FLT Model	SSG Model	Experimental Data
b_{11}^{∞}	0.209	0.202	0.208	0.219	0.203
b_{12}^{∞}	-0.155	-0.080	-0.146	-0.164	-0.156
b_{22}^{∞}	-0.148	-0.195	-0.144	-0.146	-0.143
b_{33}^{∞}	-0.061	0.007	-0.064	-0.073	-0.06
$(P/\varepsilon)_{\infty}$	1.88	3.42	1.99	1.88	1.73
$(SK/\varepsilon)_{\infty}$	6.08	21.35	6.84	5.76	5.54

Table 2. Comparison of the model predictions for the equilibrium values in homogeneous shear flow ($\mathcal{P}/\varepsilon = 1.88$) with the experimental data given in Tavoularis and Karnik (1989).

Note that the different $(\mathcal{P}/\varepsilon)_{\infty}$ attained are functions of the different $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ used in the models. The present form of the dissipation equation insures that the quantity $(\mathcal{P}/\varepsilon)_{\infty} = 1.88$ regardless of initial conditions for all Sk/ε. This is a shortcoming of the modeled dissipation equation and shows up in a larger b_{11} than the flow for which it was calibrated in which $(\mathcal{P}/\varepsilon)_{\infty} = 1.73$

Case 2: The equilibrium wall layer

Another simple but important test cases is whether the model can capture the stationary state of the log-layer in the channel flow. The homogeneous shear and the log-layer are similar in that they achieve, to a suitable approximation, an equilibrium state. Abid and Speziale (1993) have discussed the relevance of this test case and noted the inability of most rapid-pressure closures to perform successfully in the log-layer. Our results are in agreement with their contention that a model which is asymptotically consistent with the stationary states of the homogeneous shear will also do well in the log-layer. The models are compared to the DNS of the channel flow of Kim (1993) which is an update of the simulations reported in Kim *et al.* (1987). The data presented represent an average of the values of the anisotropy in the region $70 \leq y^{+} \leq 100$ outside the viscous sublayer.

Equilibrium Values	2DMFI Model	SL Model	FLT Model	SSG Model	DNS Data	Experimental Data
b_{11}^{∞}	0.180	0.079	0.141	0.201	0.180	0.22
b_{12}^{∞}	-0.141	-0.116	-0.162	-0.160	-0.134	-0.16
b_{22}^{∞}	-0.142	-0.082	-0.099	-0.127	-0.140	-0.143
b_{33}^{∞}	-0.039	0.003	-0.042	-0.074	-0.040	-0.06
$(P/\varepsilon)_{\infty}$	1.0	1.0	1.0	1.0	1.0	1.0
$(SK/\varepsilon)_{\infty}$	3.55	4.30	3.09	3.12	3.73	3.1

Table 3. Comparison of the model predictions for the equilibrium values in the log-layer of turbulent channel flow ($\mathcal{P}/\varepsilon = 1$) with the DNS data of Kim (1993) and the data of Laufer (1951) given in Abid and Speziale (1993).

Case 3: Homogeneous shear with rotation

The present test case, the homogeneous shear with rotation, and the next test case, the homogeneous shear with streamline curvature, are important test cases. In both these cases additional forces, which stabilize or destabilize the flow, are present. These effects appear in the evolution equations as additional production mechanisms for the Reynolds stresses. In the case of rotation the production terms in the evolution equation for the turbulence kinetic energy do not directly depend on the rotation: the turbulence energy production depends on the rotation only through the off-diagonal components of the Reynolds stress while in the case of streamline curvature the production terms in the evolution equation of the turbulence kinetic energy do directly dependent on the curvature. These two cases are important test cases not only because 1) the models have not been calibrated for them but also because 2) the model will have to predict both the stabilization and the destabilization of the turbulence and 3) the critical values of the governing parameters which demarcate the regions of flow stabilization from flow destabilization.

For flows in the rotating coordinate system the Coriolis terms must be carried and W_{ij} appearing in $\prod_{ij}^{r}$ must be replaced by the total rotation tensor $W_{ij} + \epsilon_{jik}\Omega_k$. Thus in the rapid-pressure model $W^* = S(1 - 2\Omega/S)$.

Figure 2b-2e show how the models perform in rotating shear for rotation to shear ratios $\Omega/S = 0, 1/4, 1/2$ compared to the LES data of Bardina *et al.* (1983) in Figure 2a. In general, all the models are able to capture both flow stabilization for some $A > \Omega/S > B$ and flow destabilization for some $A \leq \Omega/S \leq B$. The points A and B represent the points of neutral stability on a bifurcation diagram in the phase plane $(\varepsilon/Sk)_\infty$ and $(\Omega/S)_\infty$. All of the models have a bifurcation diagram of the same general form, Speziale and Mhuiris (1989), indicating a stabilization of the flow outside of some region of approximate size $0 \leq \Omega/S \leq 0.5$, predicted by the linear rapid distortion theory of Bertoglio (1982). The most important facts concerning the different models for the homogeneous rotating shear can be summarized by indicating the unstable regions in which the models predict a non-trivial equilibrium $(\varepsilon/Sk)_\infty$.

$$
\begin{aligned}
SSG: &\quad -0.09 \leq \quad \Omega/S \leq 0.53 \\
RDT: &\quad 0.00 \leq \quad \Omega/S \leq 0.50 \\
2DMFI: &\quad -0.063 \leq \quad \Omega/S \leq 0.502 \\
SL: &\quad -0.14 \leq \quad \Omega/S \leq 0.40 \\
FLT: &\quad -0.11 \leq \quad \Omega/S \leq 0.39
\end{aligned}
$$

Near the point of linear neutral stability $\Omega/S = 1/2$ both the SL and the FLT models predict a premature restabilization at values of Ω/S 20% and 22% lower than that predicted by the linear theory. The 2DMFI model is within 4% of the linear prediction.

None of the models tested, linear or nonlinear, captures the point of maximum kinetic energy growth at $(\Omega/S)_{max} = 0.25$. To do so would mean that the equations would exhibit a Richardson number similarity which, as Speziale and Mhuiris (1989) have shown, is not admitted by the Navier Stokes equations. The two models that come closest to $(\Omega/S)_{max} = 0.25$ are SSG at $\Omega/S = 0.22$, which was calibrated using this fact, and 2DMFI at $\Omega/S = 0.2$ which was not calibrated using any rotating flows. The current modeled dissipation equations predict a $(\mathcal{P}/\varepsilon)_\infty \equiv -2b_{12}Sk/\varepsilon = (C_{\varepsilon 2} - 1)/(C_{\varepsilon 1} - 1) = const.$, where the constant is model dependent but independent of rotation rate. The constant attains the value for the equilibrium homogeneous shear for arbitrary rotation rate, a fact which is not consistent with observation. In a flow that is stabilized by rotation, say $\Omega/S = 1$, production must be less than dissipation for the equilibrium state to be reached. The dissipation equation cannot be used for calibration in rotating flows without compromising the model when a dissipation equation capable of predicting the stationary values of $(\mathcal{P}/\varepsilon)_\infty = f(\Omega/S)$ becomes available.

Case 4: Homogeneous shear with curvature

For the homogeneous shear with streamline curvature the mean strain and rotation tensors are $S_{ij} = 1/2S^*(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1})$ and $W_{ij} = 1/2W^*(\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1})$ where $S^* = S(1 - stb)$ and $W^* = S(1 + stb)$ where $stb = (U_c/R_c)/S$ is the stability parameter. The geometry for the curved homogeneous shear follows that of Holloway and Tavoularis (1992): R_c is the radius of curvature of the flow and U_c is the axial velocity at the centerline and the crosstream gradient of the axial velocity is the shear $U_{1,2} = S$. The stability parameter has been renamed stb so as not to confuse it with S which is traditionally used for the mean shear. The kinetic energy growth rate is suppressed, relative to the homogeneous shear, for $stb > 0$ and increased for $stb < 0$, while for $stb > 0.05$ the experimental data indicates a relaminarization.

Figure 3 compares the model results to the experimental data for b_{12} versus stb. The plot has been generated by computing the flow from the beginning of the straight section of the wind tunnel to a value of $St = 10$ which corresponds to the end of the curved section. The initial conditions on the second order moments are given by the experimental data. The initial condition on the dissipation rate is determined by matching to the kinetic energy growth rate at the beginning of the straight section.

The different models all capture the trend in the stabilization/destabilization with respect to the stability parameter. The primary difference in the predictions of the different models seems related to their ability to capture the homogeneous shear at $stb = 0$. The results of the different models would be in more agreement for negative and small positive values of stb if they predicted the same results for the homogeneous shear.

The bifurcation diagram for the second order models, in Figure 4, was generated by letting the solution procedure go to its asymptotic state. There is a critical value of stb_c at which the stabilizing effects of curvature begins to causes a negative kinetic energy growth rate, which ultimately relaminarizes the flow. The critical values, stb_c, predicted by the different models, are

$$\begin{aligned} H\&T: &\quad stb_c = 0.05 \\ 2DMFI: &\quad stb_c = 0.067 \\ FLT: &\quad stb_c = 0.075 \\ SSG: &\quad stb_c = 0.10 \\ SL: &\quad stb_c = 0.105 \end{aligned}$$

where $H\&T$ is from the experimental data of Holloway and Tavoularis (1991). There is a consistent trend for the SSG, 2DMFI and FLT models, when compared to the critical values for the rotating shear: the higher $(\Omega/S)_c$ for stabilization of the flow correspond to higher stb_c. The SL model has small but nonzero ε/Sk over the range $0.25 < stb < stb_c$.

Case 5: Two and three-dimensional strains

The rapid pressure model is used to compute three strain flows: the plane strain and the axisymmetric contraction and expansion. These flows are another test case as the rapid pressure model has not used these flows to set the calibration coefficients. The results are compared to the direct numerical simulations of Lee and Reynolds (1985). Because the simulations are conducted at low Reynolds number the anisotropies are expected to be somewhat higher than those of a fully developed turbuelence. However the use of the physical experiments conducted at higher Reynolds number is also somewhat tenuous as the initial conditions on ε/Sk, as has been pointed out by Speziale *et al.* (1991), are not known with certainty. The same test cases as those given in Speziale *et al.* (1991) are used. Our results are also compared to the SSG model as it appears to be the current model that gives the best results. The evolution of the kinetic energy for these flows is not presented; the results for the models are in very good agreement with the data and each other and do not definitively distinguish between the various models.

Figure 5 shows the evolution of the anisotropy for the plane strain, $S_{ij} = S^*(\delta_{i1}\delta_{j1} - \delta_{i2}\delta_{j2})$ starting from isotropic initial conditions. Figure 6 and 7 shows the evolution of the anisotropy for the axisymmetric contraction and expansion. Here for the contraction $S_{ij} = S^*(\delta_{i1}\delta_{j1} - 1/2\delta_{i2}\delta_{j2} - 1/2\delta_{i3}\delta_{j3})$. For the expansion S is replacesd with $-S$. Results for all the plane and axisymmetric strain flows capture the trends nicely.

8. General discussion

The constraints of geostrophy, realizability, joint-realizability, normalization and continuity have been used to create the 2DMFI model. The use of a realizability type constraint to obtain values for unknown coefficients in the models, in the case of the strong form of realizability, has been criticized on the grounds that one should not use an extreme state to set the coefficients in an equilibrium model. This certainly is the case in any rigorous interpretation of the statement - requiring all the scales of the motion to satisfy the indicated limit. However, from the point of view of a useful engineering approximation, it should be kept in mind that $< u_i u_j >$ represents an integral over all scales of the motion: from the production scales at $\kappa\ell \sim 1$ and larger to the dissipation scales $\kappa\eta \sim 1$. In a turbulence with a $\kappa^{-5/3}$ inertial subrange in which there is enough of a separation of scales for a second-order simulation to be useful, say at least $Re_\ell \sim 10^4$, the ratio between the dissipative and the energy containing length scales is $\eta/\ell \sim Re_\ell^{-3/4} \sim 1000$ and the flow scales range over $0 < \kappa\ell < 1000$. However, approximately 85% of the energy of the motion is contained in the first decade $\kappa\ell < 10$: the major contribution to $< u_i u_j >$ is from the scales of the motion greater than one tenth of the production scales, for which $\kappa_\ell \ell \sim 1$. If only the largest 1% of the flow scales, *i.e.* from $0 < \kappa\ell < 10$, begins to lose an eigenvalue of the Reynolds stress tensor, through some dynamical or kinematical agency, $< u_i u_j >$ begins to approach the realizable limit.

Part of the reluctance to accept the second-order modeling technique is that turbulence simulations of complex inhomogeneous flows with multi-dimensional mean flows which may have body forces, streamline curvature or rotational effects, are extremely difficult to compute. ¿From a strictly practical point of view - *computability* - incorporating the realizability constraints into the models for unknown correlations has some very tangible and beneficial effects. During the convergence to a solution, from more or less arbitrary initial conditions, the iteration will be plagued with negative normal stresses and correlation coefficients larger than one. When this occurrs the solution is clipped and the solution procedure restarted from the new clipped initial conditions. The frequency of this clipping and resetting procedure is substantially reduced (in simple flows even eliminated) when using realizable models.

How one gets to a realizable turbulence is not an issue in problems with steady states, as long as the final state is realizable. This, however, is not the case for problems that are unsteady. For time-varying flows, varying on the integral time-scale, the quasi-steady problem for which the second-order methods are still suitable - this issue becomes serious. Our experience with the Reynolds averaging procedure, in buoyantly driven elliptic flows with rotation, indicates that it acts as a low-pass filter: the rapidly fluctuating instantaneous dynamics are subsumed by the averaging procedure leaving the slow-time large-scale parts of the flow, evolving on time scales commmensurate with the integral time scale, to be captured by the computation. If the simulation is to reflect the physics of the time evolution of the flow, it must stay realizable. Clearly excessive realizability violations, requiring a clipping and resetting of the solution, which produces a solution that never evolves far from the transient associated with the most recent clipped initial condition, are not acceptable. In such flows, satisfying the realizability constraint has very important consequences regarding the validity of the time evolution of the flow.

The coefficients of the tensor polynomials used to represent an unknown correlation in a constitutive relation are, according to the theory, *non-constant* functions of the invariants of the independent tensors and thus depend on the state of the turbulence. These coefficients, in "realizable" turbulence models using some form of the strong realizability constraint, are obtained by requiring that the rate of change of a positive semi-definite quantity be zero when some limit state is achieved. The coefficients so obtained are *constants* and are strictly valid *only* at the limit state. To the limit state value of the coefficients, which are regarded as a zeroeth order approximation to the coefficient in a general flow, *ad hoc* corrections are then added. These corrections, which vanish as the limit state is approached, require some sort of numerical optimization with well known simple flows. This summarizes the methodology used in other "realizable" turbulence models. In the linear SSG model the coefficients are, for the most part, constants that are *also* set by matching to a limit state. In the case of the SSG model the structural equilibrium limit is used to set the constants in the model. This is a much closer approximation to the turbulence expected to be seen in engineering problems. The constants in the SSG model may be viewed as the values the nonconstant coefficients have near the equilibrium state.

In the present method the realizability principles are used to obtain *non-constant* values of the coefficients in the constitutive relations that are valid for all states of the mechanical turbulence. Recall that the basic form of the rapid-pressure model is comprised of two parts X^0_{ijkl} and FX^F_{ijkl}, the "off-realizability" correction. X^0_{ijkl} satisfies simultaneously the five constraints - geostrophy, realizability, joint-realizability, normalization and continuity, while FX^F_{ijkl}, also obtained analytically, satisfies the substantially less extreme joint-realizability as well as the homogeneous form of the normalization and continuity constraints. Thus, although the model is consistent with the realizable limit state, the values of the model coefficients are not the values the coefficients have at the limit state. An extreme state of the flow has only been used to set the coefficients in the X^0_{ijkl} part of the model - the additional FX^F_{ijkl} terms are, for the mechanical turbulence, fully general.

It is to this basic model, valid for all states of turbulence not just at the realizability limit, that one adds the FX^∞_{ijkl} term that is necessary to obtain the equilibrium values in the flows evolution from arbitrary initial conditions. The requirement of asymptotic consistency with an equilibrium state, first used by Speziale *et al.* (1991), is the single most important and *consistent* physical requirement one can impose. Second-order closure methodology is built around the assumption that there is, to a suitable approximation, for the class of flows to which second-order methods are appropriate, a equilibrium state and in the absence of disturbing forces the flow relaxes to that state on a time scale similar to the eddy turnover time. It is precisely this phenomenological behavior that is built into the model by requiring the fixed points of the modeled equations to be consistent with those obtained from experiment. The assumption that allows the parameterization of the two-point correlation as a local function of the anisotropy tensor, $X_{ijkl} = X_{ijkl}(b_{ij})$, is such an equilibrium assumption.

The penalty paid for these additional features associated with the satisfaction of the mathematical constraints is a more complex model. It should, however, be pointed out that the present 2DMFI model has the same tensor basis as the FLT model and is therefore no more complex except for the expressions for the nonconstant coefficients. Moreover, the penalty is slight in the light of the reduction of the computational difficulties found during the calculation of quasi-steady time evolving flows with this representation for the rapid-pressure. The model, along with several other models, has been used to compute inhomogeneous buoyancy driven rotating flows that occur in the Czochralski crystal growth melt in which the Reynolds stress are three-dimensional, Ristorcelli and Lumley (1991a, 1993), Ristorcelli (1991). In computing these time varying flows it was found that the present 2DMFI model produced far fewer realizability violations during the course of the flows evolution. For a quasi-steady flow this is a crucial point: every time realizability is violated the solution is reset and the solution never evolves past the transients associated with reseting the initial conditions. Such a computation cannot be expected to reflect an ensemble average of the original system.

9. Suggestions for future work

In the effort to produce a representation for the rapid-pressure correlation valid for the class of flows to which second-order modeling is suitable, some shortcomings in the data on homogeneous "building block" flows have become apparent. Though the homogeneous shear seems reasonably well-documented it is not clear that the asymptotic states have been reached in some of the experiments. Moreover the discrepancy between the high values of b_{11}^{∞} obtained in the DNS versus those seen in the laboratory data has not been explained. Additional work expanding on the notion of two classes of flows, according to Tavoularis and Karnik (1989), and therefore the possibility of two equilibrium states, merits investigation.

For the homogeneous shear with rotation, a very basic flow, there doesn't seem to be substantial data, LES, DNS or experimental definitively describing its stationary states. At the very least an assessment of the bifurcation diagram $(\varepsilon/Sk)_{\infty}$ versus $(\Omega/S)_{\infty}$ predicted by the linear theory would be useful. Also the equilibrium values $(\mathcal{P}/\varepsilon)_{\infty}$ would be useful for further developments regarding the dissipation equation's dependence on rotation. This may remedy the under-prediction of the kinetic energy growth rates as a function of Ω/S for all the models. The present class of dissipation equations predicts an asymptotic state in which $(\mathcal{P}/\varepsilon)_{\infty}$ is a model dependent constant, which for all rotation rates has the same value that it does in the asymptotic shear. Had the stationary values of the anisotropy tensor and $(\mathcal{P}/\varepsilon)_{\infty}$ and $(\varepsilon/Sk)_{\infty}$ been available application of the present methodology would have produced a set of modeled evolution equations whose fixed points matched to the fixed points of the rotating shear, independent of the deficiencies in the dissipation equation.

The present rapid-pressure model is expected to distinguish itself in complex three-dimensional flows. For the planar flows for which test cases exist the model out-performs the nonlinear models using realizability type constraints. It is only moderately better than the topologically generic form of the SSG model suitable for simple planar flows. It is unfortunate that there does not exist any suitable DNS or LES of flows in which the presence of a body force causes the larger scales of the motion to become quasi-two-dimensional. Such a test case would help further establish the utility of incorporating some of the more complex physics into the structure of the model as well as perhaps pointing out the potential deficiencies of models developed using two-dimensional mean flows.

10. Summary and Conclusions

A new representation for the rapid-pressure strain correlation with a minimum of *ad hoc* constants has been devised. The rapid-pressure model produces the proper behavior in five different limits:

1) the geostrophic limit in which the eigenvalue of the Reynolds stress tensor, $< u_i u_j >$, along the axis of rotation vanishes,

2) the realizable limit in which an arbitrary eigenvalue of the Reynolds stress vanishes,

3) the joint-realizable limit in which an eigenvalue of $< \theta\theta >< u_i u_j > - < \theta u_i >< \theta u_j >$ vanishes,

4) the isotropic limit in which the anisotropy tensor, $b_{ij} = 0$,

5) the asymptotic limit of a structural equilibrium in which $D/Dt\ b_{ij} = 0$.

The model has the general form

$$X_{ijkl} = \quad X^0_{ijkl} + FX^F_{ijkl} + FX^\infty_{ijkl}$$

where the X_{ijkl} are polynomials in the anisotropy tensor. X^0_{ijkl} satisfies the five constraints: the limit states of geostrophy, realizability, joint-realizability, and the integral constraints of continuity and normalization. X^F_{ijkl} satisfies the three constraints of joint-realizability, continuity and normalization. Both X^0_{ijkl} and FX^F_{ijkl} are obtained analytically: they represent the simplest analytical expressions that are capable of satisfying all the mathematical constraints. X^∞_{ijkl} is obtained by requiring asymptotic consistency with the structural equilibrium state for homogeneous shear. Experimental data is required to determine X^∞_{ijkl}. The stationary values of the anisotropy tensor, b^∞_{ij}, and $(\mathcal{P}/\varepsilon)_\infty$ and $(\varepsilon/Sk)_\infty$ are inserted into the modeled evolution equations to obtain the structural equilibrium component of the rapid-pressure, X^∞_{ijkl}. This insures that the fixed points of the modeled equations match the experimentally determined fixed points. This process has been carried out for the homogeneous shear producing a model that should be accurate for three-dimensional mean flows in which the mean shear is the predominant production mechanism.

Several points regarding the 2DMFI model merit mention:

1) All the coefficients, A_i, in the $X^0_{ijkl} + FX^F_{ijkl}$ portion of the model are obtained from first principles and are valid for all states of the mechanical turbulence. They take on their values at

the limit states only when the limit states are achieved unlike other models.

2) The present form of the model is consistent with the stationary state of homogeneous shear. This insures that in the absence of any disturbing forces, the predicted flow will relax to the equilibrium state.

3) The model is at present the only model which is consistent with predictions of the Taylor-Proudman theorem assuring that the modeled equations are frame-indifferent when the components of the Reynolds stress along the axis of rotation vanish. This frame invariance is the most notable feature of the model and is expected to be important for the computation of engineering and geophysical flows in which body forces play an important role. In meteorological flows the present 2DMFI rapid-pressure correlation is consistent with the geostrophic limit atmosphere attains above the planetary boundary layer. As such the model will be relevant to the computation of mesoscale meteorological flows.

It is expected that the satisfaction of the 2DMFI principle will be important in three-dimensional flows in which the largest scales of the motion are two-dimensionalized by body forces or kinematic constraints. These flows include: 1) turbulence in which a strong stable stratification suppresses the vertical component of the velocity field; 2) turbulence in the flows affected by magnetic fields 3) turbulence influenced by rotation such as those in crystal growth processes or occurring in tornadoes, swirl combustors and turbines; 4) turbulence influenced by Coriolis forces in which the largest scales of the flow undergo a Taylor-Proudman reorganization ($Ro < 2$) as might occur in large scale geophysical flows; 5) turbulence near a free surface at which one of the components of the fluctuating velocity is suppressed and the mean flow is two-dimensional; 6) the environmentally important shallow water free shear flows as the shallow water jet associated with waste heat exchangers, near-shore pollution dispersal, tidal estuary flows, and mixing associated with thermal and salinity inflows in lakes and rivers. However, until suitable data bases, DNS or LES, of these complex flows with body forces become available the full potential of a rapid-pressure model built from first principles in three-dimensional flows can not be verified. The present model does however reproduce the experimental data at least as well as the currently available models for a wide class of two dimensional flows.

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Appendix 1: A synopsis of the final rapid-pressure correlation representation

For convenience and clarity the final form of the model is summarized here. The general final form of the model is:

$$
\begin{aligned}
\prod_{ij}^{r}/2q^2 = \;& [C_3 - 2IIC_3'' + 3IIIC_3'''] S_{ij} \\
+\;& C_4[b_{ip}S_{pj} + b_{jp}S_{pi} - 2/3 < bS > \delta_{ij}] + C_4''' < bS > [b_{ij}^2 + 2II/3\delta_{ij}] \\
+\;& C_5[b_{ip}W_{pj} + b_{jp}W_{pi}] + [C_6 < bS > + C_4''' < b^2 S >] b_{ij} \\
+\;& C_7[b_{ip}^2 S_{pj} + b_{jp}^2 S_{pi} - 2/3 < b^2 S > \delta_{ij}] \\
+\;& C_8[b_{ip}^2 W_{pj} + b_{jp}^2 W_{pi}] + C_9[b_{ip}W_{qp}b_{qj}^2 + b_{jp}W_{qp}b_{qi}^2].
\end{aligned}
$$

For flows in rotating coordinate systems, W_{ij} appearing in $\prod_{ij}^{r}$ must be replaced by the total rotation tensor $W_{ij} + \epsilon_{jik}\Omega_k$. The coefficients C_i are given by a sum of the basic model coefficients, B_i, which come from first principles, and the calibration coefficients A_i^c. The basic model coefficients are related to the coefficients A_i in the fourth order tensor polynomial expression by

$$
\begin{aligned}
B_3 =\;& 2(A_1 + A_2) = (2/27)[41 + 42II - 0.1F(221 + 420II)]/II_d \\
B_3'' =\;& A_9 + A_{10} = -(14/3)(1 + 3II)/II_d + 0.6F/(1 + 3II) \\
B_3''' =\;& -1/3(A_{11} + A_{12} + 2A_{13}) = (55 + 84II)/3II_d \\
B_4 =\;& A_3 + A_4 + 2A_5 = 3/II_d - 0.9F/(1 + 3II) \\
B_4''' =\;& A_{11} + A_{12} + 4A_{13} = -9/II_d \\
B_5 =\;& A_3 - A_4 = -(1/30)(10 + 21F)/(1 + 3II) \\
B_6 =\;& 2A_9 + 4A_{10} = -18II/II_d + 3F/(1 + 3II) \\
B_7 =\;& A_6 + A_7 + 2A_8 - 2A_9 - 2A_{10} = -9/II_d - 1.8F/(1 + 3II) \\
B_8 =\;& A_6 - A_7 = (1/5)(3F - 5)/(1 + 3II) \\
B_9 =\;& A_{11} - A_{12} = -3/(1 + 3II)
\end{aligned}
$$

The A_i are given in Appendix 3. The C_i, which reflects the application of the additional constraint, asymptotic consistency with an equilibrium state, are obtained from the B_i by $C_i = B_i + FA_i^c$. They are given by

$$
\begin{aligned}
C_3 =\;& B_3 - 2F(10A_8^c + 3A_9^c + A_{10}^c)II/5 - F(A_{11}^c + A_{12}^c + 14A_{13}^c))III/5 \\
C_3'' =\;& B_3'' + F(A_9^c + A_{10}^c) \\
C_3''' =\;& B_3''' - 1/3F(A_{11}^c + A_{12}^c + 2A_{13}^c) \\
C_4 =\;& B_4 + F(-3A_5^c + II(A_{11}^c + A_{12}^c + 4A_{13}^c)) \\
C_4''' =\;& B_4''' + F(A_{11}^c + A_{12}^c + 4A_{13}^c) \\
C_5 =\;& B_5 + F(-7/3A_5^c + (-A_{11}^c + 3A_{12}^c + 4A_{13}^c)II/3) \\
C_6 =\;& B_6 + F(2A_9^c + 4A_{10}^c) \\
C_7 =\;& B_7 - 3F(A_8^c + A_9^c + A_{10}^c) \\
C_8 =\;& B_8 - 1/3F(7A_8^c + 3A_9^c - A_{10}^c) \\
C_9 =\;& B_9 + F(A_{11}^c - A_{12}^c)
\end{aligned}
$$

where the A_i^c are expressed in terms of the seven free parameters $A_i^c\{5, 8-13\}$. Without further specifying the calibration coefficients, A_i^c, the above rapid-pressure model is general, suitable for the flows for which second-order modeling is suitable. Calibration to a particular archetypal flow, by matching the fixed points of the modeled evolution equations with experimentally determined fixed points, will result in a model suitable for diverse flows within that class of flows. For a flow in which the dominant production mechanism is associated with the mean shear the hommogeneous shear is used to set the model coefficients. Asymptotic consistency with the homogeneous equilibrium shear and a linear model for the return coefficient produce the following values for the calibration coefficients:

$$\begin{aligned} A_5^c &= -0.29 - 0.06(A_{10}^c - A_8^c) \\ A_{11}^c &= -3.6 + 5A_{10}^c - 2A_{13}^c - 12.7A_8^c - 3.8A_9^c \\ A_{12}^c &= -24.5 - 44.2A_{10}^c - 2A_{13}^c + 29A_8^c - 8A_9^c \end{aligned}$$

where $A_8^c = 0.8$, $A_9^c = -1.0$, $A_{10}^c = 0.01$, $A_{13}^c = 0$.

The homogeneous form of the continuity and normalization conditions have been used to produce relations between the free parameters. They are:

$$\begin{aligned} &A_1^c + 4A_2^c - 2A_8^c II + III(A_{11}^c + A_{12}^c + 2A_{13}^c) = 0 \\ &A_3^c + A_4^c + 5A_5^c - II(A_{11}^c + A_{12}^c + 4A_{13}^c) = 0 \\ &A_6^c + A_7^c + 5A_8^c + A_9^c + A_{10}^c = 0 \end{aligned}$$

$$\begin{aligned} &3A_1^c + 2A_2^c - 2A_6^c II + 4A_{13}^c III = 0 \\ &3A_4^c + 4A_5^c - 2II(A_{11}^c + 2A_{13}^c) = 0 \\ &3A_7^c + 4A_8^c + 2A_{10}^c = 0 \end{aligned}$$

The fact that the A_i^c must satisfy these equations has been used to express six of the $A_i^c\{1-4,6,7\}$ in terms of the seven $A_i^c\{5,8-13\}$ in the expressions for the $C_i = B_i + FA_i^c$.

Appendix 2: The calibration coefficients in the rapid-pressure representation

The model has been calibrated to match the fixed points of the homogeneous shear. The calibration process is described more detail in this appendix. The following equations describe the evolution of the anisotropy:

$$\begin{aligned} D/Dt\ b_{ij} = \ & -2\epsilon_{ikp}b_{pj}\Omega_k - 2\epsilon_{jkp}b_{pi}\Omega_k - [b_{ip}U_{j,p} + b_{jp}U_{i,p} - 2/3\delta_{ij} < bS >] \\ & -2/3S_{ij} + 2b_{ij} < bS > + \textstyle\prod_{ij}^r/q^2 - (C_1 - 2)b_{ij}\varepsilon/q^2 + C_2[b_{ij}^2 + 2II/3\delta_{ij}]\varepsilon/q^2 \end{aligned}$$

For the planar flow case with axial mean flow and mean shear, $U_{1,2}$, the algebraic equations for the fixed points, setting $D/Dtb_{ij} = 0$, become

$$\begin{aligned} &(b_{11} + 1/3)(2 + 4b_{12}Sk/\varepsilon) - 4b_{12}Sk/\varepsilon + 4(Sk/\varepsilon)\textstyle\prod_{11}^r/(4kS) - C_1 b_{11} - 2/3 \\ &+C_2(b_{11}b_{11} + b_{12}b_{12} + 2II/3) - 8(\Omega/S)(Sk\varepsilon)b_{12} = 0 \end{aligned}$$

$$\begin{aligned} &b_{12}(2 + 4b_{12}Sk/\varepsilon) - 2(b_{22} + 1/3)Sk/\varepsilon + 4(Sk/\varepsilon)\textstyle\prod_{12}^r/(4kS) - C_1 b_{12} \\ &+C_2 b_{12}(b_{11} + b_{22}) + 4(\Omega/S)(Sk/\varepsilon)(b_{11} - b_{22}) = 0 \end{aligned}$$

$$\begin{aligned} &(b_{22} + 1/3)(2 + 4b_{12}Sk/\varepsilon) + 4(Sk/\varepsilon)\textstyle\prod_{22}^r/(4kS) - C_1 b_{22} - 2/3 \\ &+C_2(b_{22}b_{22} + b_{12}b12 + 2II/3) + 8(\Omega/S)(Sk/\varepsilon)b_{12} = 0 \end{aligned}$$

$$(b_{33} + 1/3)(2 + 4b_{12}Sk/\varepsilon) + 4(Sk/\varepsilon)\textstyle\prod_{33}^r/(4Sk) - C_1 b_{33} - 2/3 + C_2(b_{33}b_{33} + 2II/3) = 0$$

The last equation for b_{33} is not linearly independent as $b_{jj} = 0$. The following fixed point values are taken to be representative: $b_{11}^\infty = 0.203$, $b_{12}^\infty = -0.156$, $b_{22}^\infty = -0.143$, $b_{33}^\infty = -0.06$, $(Sk/\varepsilon)_\infty = 5.54$,

$(P/\varepsilon)_\infty = 1.73.$ are inserted into the algebraic equations above. The values of the invariants corresponding to these values of the anisotropy tensor are: $II_\infty = -0.0604, III_\infty = 0.0038, F_\infty = 0.5613$. The solution of the algebraic equations describing the stationary state of the homogeneous shear produces the following values for the calibration coefficients,

$$\begin{aligned}
A_5^c &= -0.3388 - 0.06(A_8^c + A_{10}^c) + 0.015C_1^\infty - 0.008C_2^\infty \\
A_{11}^c &= -14.35 - 2A_{13}^c - 12.68A_8^c - 3.80A_9^c - 5.072A_{10}^c + 3.157C_1^\infty + 0.5898C_2^\infty \\
A_{12}^c &= -16.87 - 2A_{13}^c + 28.7A_8^c - 8.05A_9^c - 44.81A_{10}^c - 12.17C_1^\infty - 0.291C_2^\infty.
\end{aligned}$$

The model has been left as general as possible - allowing for any return term of the canonical form, $-C_1\varepsilon b_{ij} + C_2[b_{ij}^2 + 2II/3\delta_{ij}]$, were the C_i are not necessarily constants but have achieved their asymptotic values $C_i = C_i^\infty$. Choosing a linear Rotta type return term setting the nonlinear return coefficient to zero, $C_2 = 0$, produces

$$\begin{aligned}
A_5^c &= -0.29 - 0.06(A_{10}^c - A_8^c) \\
A_{11}^c &= -3.6 + 5A_{10}^c - 2A_{13}^c - 12.7A_8^c - 3.8A_9^c \\
A_{12}^c &= -24.5 - 44.2A_{10}^c - 2A_{13}^c + 29A_8^c - 8A_9^c
\end{aligned}$$

where the following values of the free parameters have been chosen: $A_8^c = 0.8, A_9^c = -1.0, A_{10}^c = 0.01, A_{13}^c = 0$. These indicated calibration coefficients are appropriate for the class of flows in which the mean shear is the predominant production mechanism.

The component form of the rapid-pressure used in the equations for the stationary state is

$$\begin{aligned}
\textstyle\prod_{11}^r /2q^2S^* = \;& C_4b_{12}/3 + C_4''''(b_{11}b_{11} + b_{12}b_{12} + 2II/3)b_{12} + C_5b_{12}(-W^*/S^*) \\
&+[C_6 + C_4''''(b_{11} + b_{22})]b_{11}b_{12} + C_7(b_{11} + b_{22})b_{12}/3 \\
&+C_8(b_{11} + b_{22})b_{12}(-W^*/S^*) + C_9(W^*/S^*)b_{12}(b_{12}b_{12} - b_{11}b_{22})
\end{aligned}$$

$$\begin{aligned}
\textstyle\prod_{22}^r /2q^2S^* = \;& C_4b_{12}/3 + C_4''''(b_{22}b_{22} + b_{12}b_{12} + 2II/3)b_{12} + C_5b_{12}(W^*/S^*) \\
&+[C_6 + C_4''''(b_{11} + b_{22})]b_{22}b_{12} + C_7(b_{11} + b_{22})b_{12}/3 \\
&+C_8(b_{11} + b_{22})b_{12}(W^*/S^*) + C_9(-W^*/S^*)b_{12}(b_{12}b_{12} - b_{11}b_{22})
\end{aligned}$$

$$\begin{aligned}
\textstyle\prod_{12}^r /2q^2S^* = \;& 1/2[C_3 - 2IIC_3'' + 3IIIC_3'''] + 1/2C_4(b_{11} + b_{22}) + C_4''''(b_{11} + b_{22})b_{12}b_{12} \\
&+C_5(b_{11} - b_{22})(W^*/2S^*) + [C_6 + C_4''''(b_{11} + b_{22})]b_{12}b_{12} \\
&+1/2C_7(b_{11}b_{11} + 2b_{12}b_{12} + b_{22}b_{22}) \\
&+C_8(b_{11}b_{11} - b_{22}b_{22})(W^*/2S^*) + C_9(b_{11}b_{22} - b_{12}b_{12})(b_{11} - b_{22})(W^*/2S^*)
\end{aligned}$$

$$\begin{aligned}
\textstyle\prod_{33}^r /2q^2S^* = \;& -2/3C_4b_{12} + C_4''''(b_{33}b_{33} + 2II/3)b_{12} + (C_6 + C_4''''(b_{11} + b_{22}))b_{12}b_{33} \\
&-2/3C_7b_{12}(b_{11} + b_{22}).
\end{aligned}$$

Appendix 3: The representation of the integral of the two-point covariance

The general form of the fourth-order tensor polynomial used to model the volume integral of the derivative of the two-point covariance, X_{ijkl}, appearing in the Reynolds stress equation is

$$\begin{aligned} X_{ijkl}/ < u_p u_p >= \; & A_1\delta_{ij}\delta_{kl} + A_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ + \; & A_3\delta_{ij}b_{kl} + A_4 b_{ij}\delta_{kl} + A_5(b_{ik}\delta_{jl} + b_{il}\delta_{jk} + \delta_{ik}b_{jl} + \delta_{il}b_{jk}) \\ + \; & A_6\delta_{ij}b^2_{kl} + A_7 b^2_{ij}\delta_{kl} + A_8(b^2_{ik}\delta_{jl} + b^2_{il}\delta_{jk} + \delta_{ik}b^2_{jl} + \delta_{il}b^2_{jk}) \\ + \; & A_9 b_{ij}b_{kl} + A_{10}(b_{ik}b_{jl} + b_{il}b_{jk}) \\ + \; & A_{11}b_{ij}b^2_{kl} + A_{12}b^2_{ij}b_{kl} + A_{13}(b^2_{ik}b_{jl} + b^2_{il}b_{jk} + b_{ik}b^2_{jl} + b_{il}b^2_{jk}) \\ + \; & A_{14}b^2_{ij}b^2 kl + A_{15}(b^2_{ik}b^2_{jl} + b^2_{il}b^2_{jk}) \end{aligned}$$

The coefficients with all free parameters set to zero, as derived by Ristorcelli (1991) or in Ristorcelli and Lumley (1987,1991), are

$$\begin{aligned} A_1 = \; & (111II + 73)/27II_d - F(420II + 239)/135II_d \\ A_2 = \; & -(69II + 32)/27II_d + F(420II + 257)/270II_d \\ A_3 = \; & (3II + 4)/3II_d - F(11/10)/(1 + 3II) \\ A_4 = \; & (15II + 11)/3II_d - F(4/10)/(1 + 3II) \\ A_5 = \; & -3(1 + 3II)/3II_d + F(3/10)/(1 + 3II) \\ A_6 = \; & -(102II + 61)/3II_d \\ A_7 = \; & -2(33II + 20)/3II_d - F(6/10)/(1 + 3II) \\ A_8 = \; & (42II + 23)/3II_d \\ A_9 = \; & -(57II + 28)/3II_d - F(3/10)/(1 + 3II) \\ A_{10} = \; & (15II + 14)/3II_d + F(9/10)/(1 + 3II) \\ A_{11} = \; & -(102II + 61)/II_d \\ A_{12} = \; & -2(33II + 20)/II_d \\ A_{13} = \; & (42II + 23)/II_d \\ A_{14} = \; & 0 \\ A_{15} = \; & 0 \end{aligned}$$

where $II_d = (1+3II)(7+12II)$, $F = 1+27III+9II$, where $II = -1/2b_{ij}b_{ij}$ and $III = 1/3b_{ip}b_{pj}b_{ji}$. The rapid-pressure correlation integral, X_{pkj}, appearing in the heat-flux equations, is used to derive the A_i through the joint-realizability constraint. The general form of the third-order tensor polynomial used to model X_{pkj} is included for completeness:

$$\begin{aligned} X_{pkj} = \; & D_1 < \theta u_p > \delta_{kj} + D_2[< \theta u_k > \delta_{pj} + < \theta u_j > \delta_{pk}] \\ + \; & D_3 < \theta u_p > b_{kj} + D_4[< \theta u_k > b_{pj} + < \theta u_j > b_{pk}] \\ + \; & D_5 < \theta u_p > b^2_{kj} + D_6[< \theta u_k > b^2_{pj} + < \theta u_j > b^2_{pk}] \\ \\ + \; & [D_7 b_{qp}\delta_{kj} + D_8(b_{qk}\delta_{pj} + b_{qj}\delta_{pk})] < \theta u_q > \\ + \; & [D_9 b_{qp}b_{kj} + D_{10}(b_{qk}b_{pj} + b_{qj}b_{pk})] < \theta u_q > \\ + \; & [D_{11}b_{qp}b^2_{kj} + D_{12}(b_{qk}b^2_{pj} + b_{qj}b^2_{pk})] < \theta u_q > \\ \\ + \; & [D_{13}b^2_{qp}\delta_{kj} + D_{14}(b^2_{qk}\delta_{pj} + b^2_{qj}\delta_{pk})] < \theta u_q > \\ + \; & [D_{15}b^2_{qp}b_{kj} + D_{16}(b^2_{qk}b_{pj} + b^2_{qj}b_{pk})] < \theta u_q > \\ + \; & [D_{17}b^2_{qp}b^2_{kj} + D_{18}(b^2_{qk}b^2_{pj} + b^2_{qj}b^2_{pk})] < \theta u_q > \end{aligned}$$

Where the D_i are given in Ristorcelli and Lumley (1987,1991) as

$$\begin{aligned}
D_1 &= -(312II^2 + 149II - 21)/5II_d - F(1/5)/(1+3II) \\
D_2 &= (48II^2 + II - 14)/5II_d + F(3/10)/(1+3II) \\
D_3 &= -(324II^2 + 222II + 17)/II_d - F(3/10)/(1+3II) \\
D_4 &= -3/(7+12II) + F(9/10)/(1+3II) \\
D_5 &= -(102II + 61)/II_d \\
D_6 &= (42II + 23)/II_d \\
D_7 &= -2(3II + 4)/5II_d - F(3/5)/(1+3II) \\
D_8 &= 27(2II + 1)/5II_d \\
D_9 &= (42II + 23)/II_d \\
D_{10} &= (42II + 23)/II_d \\
D_{13} &= -8(39II + 22)/5II_d \\
D_{14} &= 2(24II + 17)/5II_d \\
D_{15} &= -27/(1+3II)
\end{aligned}$$

and $D_{11} = D_{12} = D_{16} = D_{17} = D_{18} = 0$. The first term of each of the coefficients A_i, D_i satisfies the constraints of continuity, normalization, realizability, joint-realizability and 2DMFI. The terms proportional to F are the terms that contribute to the rapid-pressure when the turbulence is away from both the realizable and the 2DMFI limits.

Appendix 4: The Cayley-Hamilton theorem generalization

Reference has been made to a generalized Cayley-Hamilton theorem, Rivlin (1955), relating different powers of products of matrices.

$$\begin{aligned}
ABC \quad & +ACB + BCA + BAC + CAB + CBA = \\
& A(<BC> - <B><C>) + B(<CA> - <C><A>) + C(<AB> - <A><B>) \\
& +(BC + CB) <A> +(CA + AC) <B> +(AB + BA) <C> \\
& +1[<A><B><C> - <A><BC> - <B><AC> - <C><AB> \\
& + <ABC> + <CBA>]
\end{aligned}$$

The Cayley-Hamilton generalization is easily derivable from the Cayley-Hamilton theorem applied to sums and differences of matrices A, B, C. The theorem is useful in eliminating redundant tensor bases in tensor representation theorems. Here $<>$ is used to indicate the trace. The Cayley-Hamilton theorem for the anisotropy tensor is $b^3 = 1/3 < b^3 > 1 + 1/2 < b^2 > b$. The theorem can also be used to express

$$\begin{aligned}
\mathbf{bSb} &= -[\mathbf{b}^2\mathbf{S} + \mathbf{Sb}^2] + <\mathbf{bS}> \mathbf{b} + 1/2 <\mathbf{b}^2> \mathbf{S} + <\mathbf{b}^2\mathbf{S}> \mathbf{1} \\
\mathbf{bSb}^2 + \mathbf{b}^2\mathbf{Sb} &= -1/3 <\mathbf{b}^3> \mathbf{S} + <\mathbf{b}^2\mathbf{S}> \mathbf{b} + <\mathbf{bS}> \mathbf{b}^2
\end{aligned}$$

Appendix 4: The rapid-pressure models

The detailed form of the pressure-strain models referred to in this paper are as follows:

The Launder, Reece & Rodi Model

$$\Pi_{ij} = -2C_1\varepsilon b_{ij} + \frac{4}{5}KS_{ij} + C_2K(b_{ik}S_{jk} + b_{jk}S_{ik} - \frac{2}{3}b_{kl}S_{kl}\delta_{ij}) + C_3K(b_{ik}W_{jk} + b_{jk}W_{ik})$$

where

$$C_1 = 1.5,\ C_2 = 1.75,\ C_3 = 1.31$$

The Shih & Lumley Model

$$\begin{aligned}\Pi_{ij} &= -\beta\varepsilon b_{ij} + \frac{4}{5}KS_{ij} + 12\alpha_5 K(b_{ik}S_{jk} + b_{jk}S_{ik} - \frac{2}{3}b_{kl}S_{kl}\delta_{ij}) + \frac{4}{3}(2-7\alpha_5)K(b_{ik}W_{jk} + b_{jk}W_{ik}) \\ &\quad + \frac{4}{5}K(b_{il}b_{lm}S_{jm} + b_{jl}b_{lm}S_{im} - 2b_{ik}S_{kl}b_{lj} - 3b_{kl}S_{kl}b_{ij}) + \frac{4}{5}K(b_{il}b_{lm}W_{jm} + b_{jl}b_{lm}W_{im})\end{aligned}$$

where

$$\beta = 2 + \frac{F}{9}\exp(-7.77/\sqrt{Re_t})\{72/\sqrt{Re_t} + 80.1\ ln[1 + 62.4(-II + 2.3III)]\}$$

$$F = 1 + 9II + 27III, \quad II = -\frac{1}{2}b_{ij}b_{ij}, \quad III = \frac{1}{3}b_{ij}b_{jk}b_{ki}$$

$$Re_t = \frac{4}{9}\frac{K^2}{\nu\varepsilon}, \quad \alpha_5 = \frac{1}{10}\left(1 + \frac{4}{5}F^{\frac{1}{2}}\right).$$

The Fu, Launder & Tselepidakis Model

$$\begin{aligned}\Pi_{ij} &= \beta_1\varepsilon b_{ij} + \beta_2\varepsilon(b_{ik}b_{kj} - \frac{1}{3}b_{kl}b_{kl}\delta_{ij}) + \frac{4}{5}KS_{ij} + 1.2K(b_{ik}S_{jk} + b_{jk}S_{ik} - \frac{2}{3}b_{kl}S_{kl}\delta_{ij}) \\ &+ \frac{26}{15}K(b_{ik}W_{jk} + b_{jk}W_{ik}) + \frac{4}{5}K(b_{ik}b_{kl}S_{jl} + b_{jk}b_{kl}S_{il} - 2b_{ik}S_{kl}b_{lj} - 3b_{kl}S_{kl}b_{ij}) \\ &+ \frac{4}{5}K(b_{ik}b_{kl}W_{jl} + b_{jk}b_{kl}W_{il}) - \frac{14}{5}K[8II(b_{ik}W_{jk} + b_{jk}W_{ik}) + 12(b_{ik}b_{kl}W_{lm}b_{mj} + b_{jk}b_{kl}W_{lm}b_{mi})]\end{aligned}$$

where

$$\beta_1 = 120IIF^{1/2} + 2F^{1/2} - 2,\ \beta_2 = 144IIF^{1/2}$$

The Speziale, Sarkar & Gatski Model

$$\begin{aligned}\Pi_{ij} &= -(2C_1\varepsilon + C_1^*\mathcal{P})b_{ij} + C_2\varepsilon\left(b_{ik}b_{kj} - \frac{1}{3}b_{kl}b_{kl}\delta_{ij}\right) + (C_3 - C_3^* II_b^{\frac{1}{2}})KS_{ij} \\ &\quad + C_4K\left(b_{ik}S_{jk} + b_{jk}S_{ik} - \frac{2}{3}b_{kl}S_{kl}\delta_{ij}\right) + C_5K(b_{ik}W_{jk} + b_{jk}W_{ik})\end{aligned}$$

where

$$C_1 = 1.7,\ C_1^* = 1.80,\ C_2 = 4.2$$

$$C_3 = \frac{4}{5},\ C_3^* = 1.30,\ C_4 = 1.25$$

$$C_5 = 0.40,\ II_b = b_{ij}b_{ij}$$